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*DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE*

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*MATHEMATIQUE*

# Sur certaines classes de fonctions univalentes introduites par P. Montel et W. Rogosinski

par

Z. LEWANDOWSKI

*Presenté par K. BORSUK le 3 mars 1959*

**1.** Soit $\Sigma_0$ la classe des fonctions holomorphes et univalentes dans le cercle $|z|<1$. Nous désignerons par $z_0$ un nombre fixé tel que $0<|z_0|<1$ et par $S$, $S_0$, $S_p$ $(p=1,2,\ldots)$ les sous-classes de $\Sigma_0$ définies de la façon suivante: 1) $f(z)\in S$, si $f(0)=0$, $f'(0)=1$; 2) $f(z)\in S_0$ si $f(0)=0$, $f(z_0)=z_0$; 3) $f(z)\in S_p$, si $f(0)=0$, $f^{(p)}(z_0)=1$. Désignons encore par $S^*$, $S_0^*$, $S_p^*$ $(p=1,2,3\ldots)$ respectivement les sous-classes des classes $S$, $S_0$, $S_p$ de fonctions qui représentent le cercle $|z|<1$ sur des domaines étoilés par rapport à l'origine. Dans de nombreuses études sur la classe $S$ on a établi pour les fonctions de cette classe beaucoup de résultats intéressants. P. Montel ([5], p. 66) a proposé l'étude des classes de fonctions univalentes dans $|z|<1$ et autrement normées que celles de la classe $S$, qui est un cas limite des classes $S_0$ et $S_1$ lorsque $z_0\to 0$. Pour la classe $S_0^*$, W. Rogosinski ([6], p. 204) a obtenu le résultat suivant:

THÉORÈME R. *La transformée du cercle* $|z|<1$ *par toute fonction* $f(z)\in S_0^*$ *couvre le cercle* $|z|<\frac{1}{4}(1-|z_0|)^2$.

Ce résultat est le meilleur possible. Par exemple, la fonction $f(z)=(1-|z_0|)^2\cdot\dfrac{z}{(1-e^{-i\theta}z)^2}\in S_0^*$, où $\theta=\arg z_0$, est une fonction extrémale. Pour la classe $S_1^*$, M. Biernacki [2] a établi le théorème suivant:

THÉORÈME B. *Si* $|z_0|<\frac{1}{10}$ *et* $f(z)\in S_1^*$, *la transformée du cercle-unité par la fonction* $f(z)$ *couvre le cercle*

$$|z|<\frac{1}{4}\cdot\frac{(1-|z_0|)^3}{1+|z_0|}.$$

*La fonction* $f(z)=\dfrac{(1-|z_0|)^3}{1+|z_0|}\cdot\dfrac{z}{(1-e^{-i\theta})^2}$, *où* $\theta=\arg z_0$, *est une fonction extrémale.*

Dans le cas limite, où $z_0 \to 0$, les Théorèmes R et B fournissent les limitations connues relatives à la classe $S$.

**2.** Dans cet article je vais établir les théorèmes généralisant les résultats de W. Rogosinski et M. Biernacki, et aussi quelques résultats relatifs à la classe $S_0^*$. En particulier, je généralise un résultat de A. Marx [4] concernant la classe $S^*$.

THÉORÈME 1. *La transformée du cercle* $|z| < 1$ *par toute fonction* $\varphi(z) \in S_0$ *couvre le cercle* $|z| < \frac{1}{4}(1-|z_0|)^2$. *Cette limite est atteinte pour la fonction* $\varphi(z) = (1-|z_0|)^2 \cdot \frac{z}{(1-e^{-i\theta}z)^2}$, *où* $\theta = \arg z_0$.

Démonstration. Si $f(z) \in S$, on voit aisément que $\frac{z_0 f(z)}{f(z_0)} = \varphi(z) \in S_0$. Réciproquement, si $\varphi(z) \in S_0$, il existe une fonction $f(z) \in S$ telle, que $\varphi(z) = \frac{z_0 f(z)}{f(z_0)}$. En effet, il suffit de poser $f(z) = \frac{\varphi(z)}{\varphi'(0)}$. Les limitations connues du module des fonctions de la classe $S$ permettent de conclure que

$$\text{si } \varphi(z) \in S_0, \text{ on a } (1-|z_0|)^2 \cdot \frac{|z|}{(1+|z|)^2} \leqslant |\varphi(z)| \leqslant (1+|z_0|)^2 \cdot \frac{|z|}{(1-|z|)^2}.$$

Pour $|z| = 1$, le premier membre de cette inégalité prend la forme:

$$|\varphi(z)| \geqslant \tfrac{1}{4}(1-|z_0|)^2,$$

ce qui achève la démonstration.

THÉORÈME 2. *La transformée du cercle* $|z| < 1$ *par toute fonction de la classe* $S_1$ *couvre le cercle:* $|z| < \frac{1}{4}\frac{(1-|z_0|)^3}{1+|z_0|}$. *Cette limite est atteinte pour la fonction* $\varphi(z) = \frac{(1-|z_0|)^3}{1+|z_0|} \cdot \frac{z}{(1-e^{-i\theta}z)^2}$, *où* $\theta = \arg z_0$.

Démonstration. Si $f(z) \in S$, on voit aisément que $\frac{f(z)}{f'(z_0)} = \varphi(z) \in S_1$. Réciproquement, si $\varphi(z) \in S_1$, il existe une fonction $f(z) \in S$ telle, que $\varphi(z) = \frac{f(z)}{f'(z_0)}$. En effet, il suffit de mettre: $f(z) = \frac{\varphi(z)}{\varphi'(0)} \in S$. Les théorèmes connus sur la déformation pour la classe $S$ permettent de conclure que:

$$\frac{(1-|z_0|)^3}{1+|z_0|} \cdot \frac{|z|}{(1+|z|)^2} \leqslant |\varphi(z)| \leqslant \frac{(1+|z_0|)^3}{1-|z_0|} \cdot \frac{|z|}{(1-|z|)^2}.$$

Pour $|z| = 1$, le premier membre de cette inégalité prend la forme $|\varphi(z)| \geqslant \frac{1}{4}\frac{(1-|z_0|)^3}{1+|z_0|}$ et le Théorème 2 se trouve ainsi démontré.

Pour les fonctions $f(z) \in S$ qui représentent le cercle $|z| < 1$ sur des domaines convexes, on a les limitations connues ([5], p. 62):

$$(1) \qquad \frac{|z|}{1+|z|} \leqslant |f(z)| \leqslant \frac{|z|}{1-|z|}; \qquad \frac{1}{(1+|z|)^2} \leqslant |f'(z)| \leqslant \frac{1}{(1-|z|)^2}.$$

Si $\varphi(z) \in S_0$ et si $\varphi(z)$ représente le cercle $|z|<1$ sur un domaine convexe, il existe une fonction convexe $f(z) \in S$ telle, que $\varphi(z)=\frac{z_0 f(z)}{f(z_0)}$. Nous avons donc:

$$(1-|z_0|)\cdot\frac{|z|}{1+|z|} \leqslant |\varphi(z)| \leqslant (1+|z_0|)\cdot\frac{|z|}{1-|z|}.$$

En mettant $|z|=1$ dans le premier membre de cette inégalité nous obtenons:

THÉORÈME 1a. *La transformée du cercle* $|z|<1$ *par toute fonction* $\varphi(z) \in S_0$, *qui représente ce cercle sur un domaine convexe, couvre le cercle* $|z|<\frac{1}{2}(1-|z_0|)$. *Cette limite est atteinte pour la fonction* $\varphi(z)=(1-|z_0|) \times \frac{z}{1-e^{-i\theta}z}$, $\theta=\arg z_0$.

Si $\varphi(z) \in S_1$ et si $\varphi(z)$ est une fonction convexe, on a $\varphi(z)=\frac{f(z)}{f'(z_0)}$, où $f(z)$ est une fonction convexe appartenant à la classe $S$. En tenant compte de (1) nous obtenons:

$$\frac{|z|}{1+|z|}\cdot(1-|z_0|)^2 \leqslant |\varphi(z)| \leqslant \frac{|z|}{1-|z|}\cdot(1+|z_0|)^2.$$

Pour $|z|=1$ on a $|\varphi(z)| \geqslant \frac{1}{2}(1-|z_0|)^2$, donc:

THÉORÈME 2a. *La transformée du cercle* $|z|<1$ *par toute fonction* $\varphi(z) \in S_1$, *qui représente ce cercle sur un domaine convexe, couvre le cercle* $|z|<\frac{1}{2}(1-|z_0|)^2$. *Cette limite est atteinte pour la fonction* $\varphi(z)=(1-|z_0|)^2 \times \frac{z}{1-e^{-i\theta}z}$, $\theta=\arg z_0$.

Je démontrerai encore la proposition suivante:

THÉORÈME 3. *Si l'hypothèse de Bieberbach:* $|a_n| \leqslant n$ *pour* $n=2,3,\ldots p$ *est vérifiée pour toute fonction* $f(z) \in S$, $f(z)=z+a_2z^2+\ldots$, *la transformée du cercle* $|z|<1$ *par une fonction quelconque de la classe* $S_p$ *couvre le cercle*

$$|z|<\frac{1}{4}\frac{(1-|z_0|)^{p+2}}{p!\,(p+|z_0|)}.$$

*Cette limite est atteinte pour la fonction* $\varphi(z)=\frac{(1-|z_0|)^{p+2}}{p!\,(p+|z_0|)}\cdot\frac{ze^{(p-1)i\theta}}{(1-e^{-i\theta}z)^2}$, $\theta=\arg z_0$.

Démonstration. Si $\varphi(z) \in S_p$, il existe une fonction $f(z) \in S$ telle, que $\varphi(z)=\frac{f(z)}{f^{(p)}(z_0)}$. En effet, il suffit d'admettre $f(z)=\frac{\varphi(z)}{\varphi'(0)}$. Réciproquement, pour toute fonction $f(z) \in S$ la fonction $\varphi(z)=\frac{f(z)}{f^{(p)}(z_0)}$ appartient à la classe $S_p$. F. Marty [3] a démontré que si l'hypothèse de

Bieberbach: $|a_n| \leqslant n$, $n = 2, 3, \ldots p$, est vérifiée pour toutes les fonctions $f(z) \in S$, $f(z) = z + a_2 z^2 + \ldots$, alors

$$|f^{(p)}(z)| \leqslant p! \frac{p+|z|}{(1-|z|)^{p+2}}.$$

Il en résulte, pour le module de $\varphi(z)$, la limitation inférieure suivante:

$$|\varphi(z)| \geqslant \frac{|z|}{(1+|z|)^2} \cdot \frac{(1-|z_0|)^{p+2}}{p!\,(p+|z_0|)}$$

et, pour $|z| = 1$, on obtient ainsi le Théorème 3.

Pour terminer, je vais énoncer sans démonstration les théorèmes relatifs à la classe $S_0^*$.

THÉORÈME 4. *L'ensemble des valeurs de la fonctionnelle* $\sqrt{\frac{z}{\varphi(z)}}$, *où* $z$ *est fixé,* $|z| < 1$, *dans la classe* $S_0^*$ *est le cercle fermé:*

$$\left| \sqrt{\frac{z}{\varphi(z)}} - \frac{1-z\bar{z}_0}{1-|z_0|^2} \right| \leqslant \frac{|z-z_0|}{1-|z_0|^2} \text{ *)}.$$

Si dans le Théorème 4 on fait $z_0 \to 0$, alors $S_0^* \to S^*$ et on obtient pour la classe $S^*$ le résultat de A. Marx [4]. De la démonstration du Théorème 1 il résulte, qui si $f(z) \in S^*$, on a $\varphi(z) = \frac{z_0 f(z)}{f(z_0)} \in S_0^*$, et que toute fonction $\varphi(z) \in S_0^*$ peut être représentée sous la forme $\varphi(z) = \frac{z_0 f(z)}{f(z_0)}$, où $f(z)$ est une fonction de la classe $S^*$. Le Théorème 4 concerne le domaine de variation de l'expression $\frac{z}{\varphi(z)}$ dans la classe $S_0^*$, c'est-à-dire le domaine de variation de la fonctionnelle $\frac{z f(z_0)}{z_0 f(z)}$, $z$ et $z_0$ étant fixés, dans la classe $S^*$. En échangeant les rôles des variables $z$ et $z_0$, nous tirons immédiatement du Théorème 4:

THÉORÈME 4a. *L'ensemble des valeurs de la fonctionnelle* $\sqrt{\frac{\varphi(z)}{z}}$, *où* $z$ *est fixé et* $|z| < 1$, *dans la classe* $S_0^*$, *est le cercle fermé:*

$$\left| \sqrt{\frac{\varphi(z)}{z}} - \frac{1-\bar{z}z_0}{1-|z|^2} \right| \leqslant \frac{|z-z_0|}{1-|z|^2}.$$

De théorèmes 4 et 4a on obtient immédiatement les limitations exactes du module des fonctions de la classe $S_0^*$:

*Si* $\varphi(z) \in S_0^*$, *on a:*

$$|z| \left( \frac{1-|z_0|^2}{|z-z_0|+|1-z\bar{z}_0|} \right)^2 \leqslant |\varphi(z)| \leqslant |z| \left( \frac{|z-z_0|+|1-z_0\bar{z}|}{1-|z|^2} \right)^2.$$

*) Il s'agit de la branche de $\sqrt{z/\varphi(z)}$ qui, pour $z = z_0$, est égale à 1.

Le Théorème 4 peut être démontré au moyen de la méthode qui a été récemment donnée par Ashnievic et Ulina [1]. Cette méthode s'est montrée très utile dans l'étude de certaines classes de fonctions holomorphes. La démonstration du Théorème 4 et d'autres résultats relatifs aux classes de fonctions, que nous avons considérées ici, sera exposée dans un travail plus étendu qui paraîtra dans les Annales Universitatis M. Curie-Skłodowska.

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MATHEMATICS

# On Lower Bounds for Probability Moments

by

S. TRYBUŁA

*Presented by H. STEINHAUS on March 14, 1959*

Suppose that the random variable $X$ has an absolutely continuous distribution with density function $f(x)$. The $n$-th probability moment $\pi_n$ is defined by

$$\pi_n = \int_{-\infty}^{\infty} f^n(x)\,dx\,. \tag{1}$$

These functionals have attracted the attention of some authors as parameters of dispersion for the known functional form of $f(x)$ [3] and also in connection with fitting frequency curves [2]. In this note we shall give the greatest lower bound of parameter $\pi_n$ for random variables with finite $k$-th central absolute moment $m_k$ $(m > 1,\ k \geqslant 1)$.

Since we can put $\int_{-\infty}^{\infty} xf(x)\,dx = 0$ without loss of generality, we have

$$f(x) \geqslant 0, \quad \int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \int_{-\infty}^{\infty} |x|^k f(x)\,dx = m_k < \infty\,. \tag{2}$$

Denote by $f_0(x)$ the following density function:

$$f_0(x) = \begin{cases} c(1-|ax|^k)^{\frac{1}{n-1}} & \text{for} \quad -\dfrac{1}{a} < x < \dfrac{1}{a}, \quad (a > 0), \\ 0 & \text{otherwise}\,. \end{cases} \tag{3}$$

For each $0 < m_k < \infty$ there exist constants $c$ and $a$ such that conditions (2) hold for $f(x) \equiv f_0(x)$. Let $f$ be some function satisfying conditions (2). We have, from Taylor's expansion,

$$f^n = f_0^n + nf_0^{n-1}(f-f_0) + \binom{n}{2} f_1^{(n-2)}(f-f_0)^2, \tag{4}$$

where $f_1$ is a value between $f$ and $f_0$. From (4) we have

$$f^n - f_0^n - nf_0^{n-1}(f-f_0) \geqslant 0\,. \tag{5}$$

Let $\pi_n^0$ be the $n$-th probability moment for density (3). If $f(x)$ and $f_0(x)$ have the same $k$-th moment $m_k$ we have

$$\pi_n - \pi_n^0 = \int_{-\infty}^{\infty} [f^n(x) - f_0^n(x)]\,dx$$

$$= \int_{-\infty}^{\infty} f^n(x) - f_0^n(x) - nc^{n-1}(1-|ax|^k)[f(x)-f_0(x)]\,dx$$

$$= \int_{-\infty}^{\infty} \{f^n(x) - f_0^n(x) - nf_0^{n-1}(x)[f(x)-f_0(x)]\}\,dx \geqslant 0\,.$$

Hence, for $f \equiv f_0$, the probability moment $\pi_n$ is the smallest. From the condition

$$c \int_{-1/a}^{1/a} (1-|ax|^k)^{\frac{1}{n-1}}\,dx = 1$$

we get

$$c = \frac{ak}{2B\left(\frac{1}{k}, \frac{n}{n-1}\right)},$$

where $B(\alpha, \beta)$ is the beta-function. Furthermore,

$$m_k = \frac{ak}{2B\left(\frac{1}{k}; \frac{n}{n-1}\right)} \int_{-1/a}^{1/a} |x|^k (1-|ax|^k)^{\frac{1}{n-1}}\,dx = \frac{1}{a^k} \frac{B\left(\frac{k+1}{k}, \frac{n}{n-1}\right)}{B\left(\frac{1}{k}, \frac{n}{n-1}\right)}, \tag{6}$$

$$\pi_n^0 = \left[\frac{ak}{2B\left(\frac{1}{k}, \frac{n}{n-1}\right)}\right]^n \int_{-1/a}^{1/a} (1-|ax|^k)^{\frac{n}{n-1}}\,dx$$

$$= \left[\frac{ak}{2B\left(\frac{1}{k}, \frac{n}{n-1}\right)}\right]^{n-1} \cdot B\left(\frac{1}{k}, \frac{2n-1}{n-1}\right) \tag{7}$$

$$= \frac{k^{n-1} B^{\frac{n-1}{k}}\left(\frac{k+1}{k}, \frac{n}{n-1}\right) B\left(\frac{1}{k}, \frac{2n-1}{n-1}\right)}{2^{n-1} B^{\frac{nk+n-1}{k}}\left(\frac{1}{k}, \frac{n}{n-1}\right)} \frac{1}{m_k^{\frac{n-1}{k}}} \leqslant \pi_n\,.$$

From (7) we have finally

$$\pi_n^k\, m_k^{n-1} \geqslant \frac{(kn)^k (n-1)^{n-1}}{(kn+n-1)^{k+n-1}} \left[\frac{k}{2B\left(\frac{1}{k}, \frac{n}{n-1}\right)}\right]^{k(n-1)}. \tag{8}$$

Formula (8) takes the form

$$\pi_n \geqslant \left(\frac{n}{4n-2}\right)^n \frac{2}{m_1^{n-1}}, \quad \text{for} \quad k=1, \tag{9}$$

and

$$\pi_n \geqslant \frac{2n}{3n-1}\left(\frac{\sqrt{\frac{n-1}{3n-1}}}{B\left(\frac{1}{2}, \frac{n}{n-1}\right)}\right)^{n-1} \frac{1}{m_2^{\frac{n-1}{2}}}, \quad \text{for} \quad k=2. \tag{10}$$

The latter inequality has been obtained by S. Moriguti [1].

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*MATHEMATICS*

# Theorem on Antipodes and Theorems on Fixed Points for a Certain Class of Multi-valued Mappings in Banach Spaces

by

A. GRANAS

*Presented by K. BORSUK on March 20, 1959*

**1. Introduction.** Denote by $E_\alpha$ an arbitrary Banach space, if the index $\alpha = \infty$, and an $n$-dimensional subspace of space $E_\infty$, if the index $\alpha$ is equal to a natural number $n$. Denote by $P_\alpha$ the space $E_\alpha$ without the point 0. If $x_0$ is a point of space $E_\alpha$ and $\varrho$ a positive number, then we denote by $V_\alpha(x_0, \varrho)$ an open full-sphere with centre $x_0$ and radius $\varrho$ and by $S_{\alpha-1}(x_0, \varrho)$ — the boundary of $V_\alpha(x_0, \varrho)$.

The mapping $\varphi$ defined on the set $A$ and assigning to a point $x \in A$ a non-empty set $\varphi(x) \subset E_\alpha$ is called *upper semicontinuous*, if the conditions $\lim_{n=\infty} x_n = x$, $\lim_{n=\infty} y_n = y$, $y_n \in \varphi(x_n)$ imply $y \in \varphi(x)$. In what follows we consider only upper semicontinuous mappings and assume their values to be *convex sets* in $E_\alpha$. The notation $\varphi: A \to E_\alpha$ denotes that $\varphi$ is an upper semicontinuous mapping defined on $A$, whose every value $\varphi(x)$ is a convex set in $E_\alpha$.

A multi-valued mapping $\Phi: A \to E_\alpha$ is called *completely continuous* on $A$ *), if its image $\Phi(A) = \underset{y \in E}{E} \underset{x \in A}{\exists} \{y \in \varphi(x)\}$ is relatively compact in $E_\alpha$ **).

A mapping $\varphi: A \to E_\alpha$ is called *a completely continuous multi-valued vector field* on the set $A$, if it can be represented in the form:

(1) $\varphi(x) = x - \Phi(x)$, $x \in A$, where $\Phi$ is completely continuous on $A$ ***).

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*) In the sequel we denote multi-valued completely continuous mappings by capital letters $\Phi, \Psi \ldots$

**) The set $X \subset E_\alpha$ is termed relatively compact in $E_\alpha$ if its closure is compact.

***) For any multi-valued functions $\varphi_1, \varphi_2 A \to E_\alpha$ and real numbers $t_1, t_2$ we understand by $t_1\varphi_1(x) + t_2\varphi_2(x) = \varphi(x)$ a function which assigns to the point $x \in A$ the set $\varphi(x)$ of points of the form $z = t_1 y_1 + t_2 y_2$, where $y_1 \in \varphi_1(x)$, $y_2 \in \varphi_2(x)$.

We say that a completely continuous field $\varphi(x) = x - \Phi(x)$, $x \in A$ *does not vanish* and we write $\varphi: A \to P_a$, if the point 0 does not belong to the set $\varphi(x)$ for any $x \in A$, i. e. $x \bar{\in} \Phi(x)$.

We say that two non-vanishing completely continuous multi-valued fields $\varphi_1, \varphi_2: A \to P_a$, $\varphi_1(x) = x - \Phi_1(x)$, $\varphi_2(x) = x - \Phi_2(x)$ are *homotopic* and we write $\varphi_1 \simeq \varphi_2$, if there exists a function $\Psi(x, t)$ defined on the product $A \times J$ ($J$ denoting the closed interval [01]) with values lying in $E_a$, which satisfies the conditions:

$1^0$ point 0 does not belong to any set $\psi(x, t) = x - \Psi(x, t)$, $x \in A$, $t \in J$,

$2^0$ $\Psi(x, 0) = \Phi_1(x)$, $\Psi(x, 1) = \Phi_2(x)$ for any $x \in A$,

$3^0$ the set $\Psi(A, J)$ is relatively compact in $E_a$.

Let $S_{a-1} = S_{a-1}(x_0, \varrho)$ be a sphere in the space $E_a$. We shall refer to the following [4]:

THEOREM. *To every completely continuous field $\varphi: S_{a-1} \to E_a$ we can assign a positive number $\gamma(\varphi, S_{a-1})$ — the characteristic of the field $\varphi$ on the sphere $S_{a-1}$ in such a way that the following conditions are satisfied:*

*$\alpha$) if the completely continuous fields $\varphi_1, \varphi_2: S_{a-1} \to P_a$ are homotopic, $\varphi_1 \simeq \varphi_2$, then their characteristics are equal, i. e. $\gamma(\varphi_1, S_{a-1}) = \gamma(\varphi_2, S_{a-1})$;*

*$\beta$) if a non-vanishing completely continuous field $\varphi_0: S_{a-1} \to P_a$ can be extended to a completely continuous field $\varphi: \overline{V}_a \to E_a$ defined on a full sphere $\overline{V}_a = \overline{V_a(x_0, \varrho)}$ and the characteristic $\gamma(\varphi_0, S_{a-1})$ of the field $\varphi_0$ is different from zero, then there exists a point $x_0 \in V_a$ whose image $\varphi(x_0)$ contains point 0, i. e. $0 \in \varphi(x_0)$.*

The theorem quoted constitutes the general criterion regarding the existence of fixed points for completely continuous multi-valued mappings. In this note we give a few particular criteria, which are an extension to the case of multi-valued mappings of the basic theorems of the Leray-Schauder theory concerning the existence of fixed points for completely continuous single-valued mappings [7], [8].

**2. Theorems on fixed points.** The following theorem is an extension to the case of multi-valued functions of the well known theorem of Rothe (cf. [8], p. 186).

THEOREM 1. *Let a multi-valued completely continuous mapping $\Phi$: $\overline{V}_\infty \to E_\infty$ carry a full sphere $\overline{V}_\infty = \overline{V_\infty(x_0, \varphi)}$ into a space $E_\infty$.*

*If for an arbitrary point $x \in S_\infty = \mathrm{Fr}(\overline{V}_\infty)$ the condition $\Phi(x) \subset \overline{V}_\infty$ is satisfied (i. e. $\Phi(S_\infty) \subset \overline{V}_\infty$), then there exists at least one fixed point of the mapping $\Phi$ i. e. $x_0 \in \Phi(x_0)$ for a certain $x_0 \in \overline{V}_\infty$.*

Proof. Without reducing the generality of our considerations we can assume that the full sphere in question is the unit full sphere in the space $E_\infty$, i. e. $\overline{V}_\infty = \overline{V_\infty(0, 1)} = \overline{V}^0_\infty$. Denote by $S^0_\infty$ the boundary of $\overline{V}^0_\infty$. Consider a completely continuous field $\varphi: \overline{V}^0_\infty \to E_\infty$ defined by the formula

$\varphi(x) = x - \Phi(x)$ and let $\varphi_0 = \varphi|S^0_\infty$ *). We can assume that $\varphi_0$: $S^0_\infty \to P_\infty$ and by $\beta$) it suffices to prove that the characteristic $\gamma(\varphi_0, S^0_\infty)$ of the field $\varphi_0$ is different from zero. For this purpose let

$$\psi(x, t) = x - t\Phi(x)$$

for an arbitrary $x \in S^0_\infty$, $0 \leqslant t \leqslant 1$.

It follows from our assumption that for an arbitrary $y^* \in \psi(x, t)$, where $x \in S^0_\infty$, $0 \leqslant t < 1$ we have

$$\|y^*\| = \|x - ty\| \geqslant \|x\| - t\|y\| > 0\,,$$

and thus $\psi$: $S^0_\infty \times J \to P_\infty$. Further, $\psi(x, 0) = x$, $\psi(x, 1) = \varphi_0(x)$ for any $x \in S^0_\infty$, i. e. the fields $\psi(x, 0) = \psi_0(x)$ and $\varphi_0(x)$ are homotopic, whence, by $\alpha$), $\gamma(\psi_0, S^0_\infty) = \gamma(\varphi_0, S^0_\infty)$. But the characteristic $\gamma(\psi_0, S^0_\infty)$ of the field $\psi_0$ is different from zero and Theorem 1 is thus proved.

Theorem 1 immediately implies the well known

THEOREM OF BOHNENBLUST AND KARLIN (cf. [2] p. 155). *If $\Phi$: $A \to A$ is a multi-valued mapping into itself of a compact convex set $A$ lying in $E_\infty$, then there exists a point $x_0 \in A$ for which $x_0 \in \Phi(x_0)$.*

Proof. Denote by $\overline{V}_\infty$ a closed full sphere in space $E_\infty$ with centre 0 and radius $\varrho > 0$ so large that $A \cap \mathrm{Fr}(\overline{V}_\infty) = \theta$. Further denote by $r$: $\overline{V}_\infty \to A$ a retraction of the sphere $\overline{V}_\infty$ onto $A$ (the existence of such a retraction can be proved in an elementary way) and let $\Phi'(x) = \Phi[r(x)]$ for any $x \in \overline{V}_\infty$. The function $\Phi'$: $\overline{V}_a \to A$ satisfies of course the conditions of Theorem 1, and thus there exists a point $x_0 \in \Phi'(x_0)$, $x_0 \in A$. We have $\Phi'(x_0) = \Phi(r(x_0)) = \Phi(x_0) \ni x_0$ and the proof is thus complete.

We shall now give a theorem, which in the finite-dimensional case of single-valued functions is known as the Kronecker-Hopf theorem **).

Let $\Phi$: $\overline{V}_a \to E_a$ be a completely continuous mapping of a sphere $\overline{V}_a$ in $E_a$. Suppose that the mapping $\Phi$ has a finite number of fixed points $x_1, x_2, \ldots, x_k \in V_a$, $x_i \in \Phi(x_i)$, $(i = 1, 2, \ldots, k)$.

Denote by $V^{(i)}_a$ a full sphere in the space $E_a$ with centre $x_i$ and radius $\varrho_i > 0$ so small that $\overline{V}^{(i)}_a \cap \overline{V}^{(j)}_a$ $(i \neq j,\ i, j = 1, 2, \ldots k)$, $\overline{V}^{(i)}_a \subset V_a$. Now, for an arbitrary $x \in \overline{V}_a$ let $\varphi(x) = x - \Phi(x)$, $\varphi_{(i)} = \varphi|S^{(i)}_{a-1}$ $(i = 1, 2, \ldots k)$, where $S^{(i)}_{a-1} = \mathrm{Fr}(\overline{V}^{(i)}_a)$. We have $\varphi_{(i)}$: $S^{(i)}_{a-1} \to P_a$, $\varphi$: $S_{a-1} \to P_a$, where $S_{a-1} = \mathrm{Fr}(V_a)$. The number $\gamma_i = \gamma(\varphi_i, S^{(i)}_a)$ equal to the characteristic of the completely continuous field $\varphi_i$ on the sphere $S^{(i)}_{a-1}$ will be termed the *index of the fixed point $x_i$ of the mapping $\Phi$.*

*) If $\varphi$: $A \to E_a$, $A_0 \subset A$, then the notation $\varphi_0 = \varphi|A$ denotes that $\varphi_0$ is a partial mapping on $A_0$.

**) Cf. [8] p. 188, for a single valued case in Banach spaces.

THEOREM $2_\alpha$. *The characteristic* $\gamma = \gamma(\varphi, S_{\alpha-1})$ *of a non-vanishing completely continuous multi-valued field* $\varphi$: $S_{\alpha-1} \to P_\alpha$, $\varphi(x) = x - \Phi(x)$ *is equal to the algebraic sum of the indices of the fixed points of the mapping* $\Phi$, *i. e. we have the formula*

$$\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_k . \tag{2}$$

The proof in the finite-dimensional case $\alpha = n$ is carried out on the basis of the Vietoris Mappings Theorem ([1], theorem 1) and the case $\alpha = \infty$ reduces to the finite-dimensional case.

**3. Theorem on antipodes.** The following theorem is an extension to multi-valued functions of the well-known theorem on antipodes [3], [6].

THEOREM $3_\alpha$. *If a non-vanishing completely continuous multi-valued field* $\varphi$: $S^0_{\alpha-1} \to P_\alpha$, *defined on a unit sphere* $S^0_{\alpha-1} \subset E_\alpha$, *is odd*

$$\varphi(x) = -\varphi(-x) , \tag{3}$$

*then its characteristic* $\gamma(\varphi, S^0_{\alpha-1})$ *is odd.*

The proof in the case $\alpha = n$ is based on the corollary 1 of paper [5]. The case $\alpha = \infty$ reduces to the finite-dimensional case in the same way as in the case of single-valued mappings [6].

The theorem quoted can be formulated in a more general form on the grounds of property $\alpha$):

THEOREM $4_\alpha$. *If a non-vanishing completely continuous multi-valued field* $\varphi$: $S^0_{\alpha-1} \to P_\alpha$ *satisfies the condition*

$$\varphi(x) \cap \lambda\varphi(-x) = \theta \quad \text{for any} \quad \lambda > 0 , \tag{4}$$

*then its characteristic* $\gamma(\varphi, S^0_{\alpha-1})$ *on the sphere* $S^0_{\alpha-1}$ *is odd* **).

Proof. For an arbitrary $(x, t) \in S^0_{\alpha-1} \times J$ let

$$\Psi(x, t) = \frac{\Phi(x)}{1+t} - \frac{t\Phi(-x)}{1+t}, \quad \psi(x, t) = x - \Psi(x, t) .$$

It follows from condition (4) that the function $\psi(x, t)$ satisfies all conditions of the definition of homotopy, and thus the field $\varphi(x) = \psi(x, 0)$ is homotopic with the odd field $\psi_0(x) = \psi(x, 1) = x - \frac{1}{2}[\Phi(x) - \Phi(-x)]$ whose characteristic is odd by Theorem $3_\alpha$. Hence, by $\alpha$), we have $\gamma(\varphi, S^0_{\alpha-1}) = \gamma(\psi, S^0_{\alpha-1})$ and Theorem $4_\alpha$ is proved.

THEOREM $5_\alpha$. *If a completely continuous multi-valued field* $\varphi$: $\overline{V}^0_\alpha \to E_\alpha$, $\varphi(x) = x - \Phi(x)$ *defined on a unit full sphere* $V^0_\alpha$ *satisfies for any* $x \in S^0_{\alpha-1}$

*) In the finite-dimensional case, i. e. for $\alpha = n$ Theorems $4_\alpha$ and $5_\alpha$ can be proved in a more general form, cf. paper [5] corollary 1 and 2.

*condition* (3) *or* (4), *then there exists a fixed point* $x_0 \in V_a^0$ *of the completely continuous mapping* $\Phi$: $V_a^0 \to E_a$, *i. e.* $x_0 \in \Phi(x_0)$.

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MATHEMATICS

# Some Theorems on Multi-Valued Mappings of Subsets of the Euclidean Space

by

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**1.** The aim of this note is to give other examples of generalization of some classical theorems concerning single-valued mappings to the case of multi-valued acyclic mappings. Many such examples are known as, for instance, a generalization of the Lefschetz fixed-point formula due to S. Eilenberg and D. Montgomery [3]; others are given in [6] and [7].

**2.** Let $X$ and $Y$ be topological spaces and $F\colon X \to Y$ – a multi-valued function from $X$ to $Y$; i. e. for every $x \in X$, a non-empty subset $F(x)$ of $Y$ is given. If $A \subset X$, then the set $F(A) = \bigcup_{a \in A} F(a)$ is called the *image* of $A$. The *graph* of $F$ is the set

$$W(F) = \{(x, y) : y \in F(x)\} \subset X \times Y.$$

The single-valued function $f\colon X \to Y$ is said to be a *selector* of $F$ if $W(f) \subset W(F)$, i. e. $f(x) \in F(x)$, for every $x \in X$.

The multi-valued function $F\colon X \to Y$ is said to be *continuous* if

$1^0$ the image of every compact subset of $X$ is compact,

$2^0$ the graph of $F$ is closed in $X \times Y$.

For compact $Y$, condition $1^0$ may be dropped; in this case condition $2^0$ is equivalent to the upper semi-continuity of $F$, when $F$ is regarded as a single-valued mapping of $X$ into the hyperspace of non-empty compact subsets of $Y$. If $F$ is single-valued, i. e. for every $x \in X$, $F(x)$ is a single point $f(x)$, the continuity of $F$ means the usual continuity of the single-valued mapping $f$.

**3.** We shall use the Vietoris-Čech homology groups with coefficients mod 2 and with *compact carriers* ([4], p. 255) of metric spaces, and we shall denote simply by $H_k(X)$ the $k$-th homology group of $X$. The space $X$ is called *$n$-acyclic* if it is non-empty and if $H_k(X) = 0$ for $1 \leqslant k \leqslant n$; it is called *acyclic*, if it is $n$-acyclic for every $n \geqslant 1$. We shall often refer to the famous

VIETORIS MAPPING THEOREM [1]. *Let $f: X \to Y$ be a continuous mapping satisfying the following conditions:*

(i) *the counter-image $f^{-1}(C)$ of every compact subset of $Y$ is compact,*

(ii) *$f^{-1}(y)$ is acyclic, for every $y \in Y$.*

*Then $f$ induces an isomorphism $f_*: H_k(X) \approx H_k(Y)$ for every $k$.*

The condition (i) provides that the Vietoris Theorem remains true for homology with compact carriers even for non-compact spaces. In this note we confine ourselves to the case of homology mod 2; we do so for the sake of simplicity though some of the results are valid for homology with other coefficient groups for which the Vietoris Theorem holds. It is known, however, that this theorem does not hold when the integers are taken as coefficients.

**4.** The multi-valued mapping $F: X \to Y$ is said to be *acyclic* if the sets $F(x)$ are acyclic for every $x \in X$.

Let $F: X \to Y$ be a multi-valued continuous acyclic mapping and let $W$ be the graph of $F$. Let $p: W \to X$, $q: W \to Y$ be the natural projections, i. e.

$$p(x, y) = x, \quad q(x, y) = y. \tag{1}$$

Then the sets $p^{-1}(x)$ are acyclic and, by the Vietoris Theorem, $p$ induces an isomorphism $p_*: H_k(W) \approx H_k(X)$. The homomorphism

$$F_* = q_* p_*^{-1}: H_k(X) \to H_k(Y)$$

is said to be *induced* by the multi-valued mapping $F$. In particular, if $F$ is single-valued, it coincides with the usual induced homomorphism of homology. More generally, if $f: X \to Y$ is a continuous selector of $F$, then $f_* = F_*$ ([7], p. 263).

**5.** The Vietoris Theorem enables us to define for multi-valued acyclic mappings many homology notions analogous to those concerning single-valued mappings with similar properties.

Let $S^n$ denote the unit sphere in the Euclidean $n$-space $E^{n+1}$. Let 0 be the origin and let $P^{n+1} = E^{n+1} \setminus (0)$. Let $F: S^n \to E^{n+1}$ be a multi-valued continuous acyclic mapping.

If $F$ does not vanish on $S^n$, i. e. if $F(S^n)$ does not contain the origin, then $F$ may be regarded as a mapping $S^n \to P^{n+1}$. Such a mapping may also be looked upon as a multi-valued non-vanishing vector field on $S^n$. The notion of the Kronecker index of $F$ may be defined as follows: Since the groups $H_n(S^n)$ and $H_n(P^{n+1})$ are both isomorphic with the coefficient group, the induced homomorphism $F_*: H_n(S^n) \to H_n(P^{n+1})$ may be identified with an integer mod 2; it is called the *index* of $F$ and is denoted by $I(F)$ *).

*) The notion of index may be likewise defined for mappings of more general subsets of $E^{n+1}$, for instance, for mappings of $n$-dimensional manifold; it may also be defined for other coefficient groups, as has already been mentioned in [4].

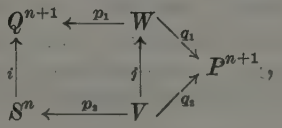
In the case, when $y_0$ is an arbitrary point of $E^{n+1}$ and $F(S^n)$ does not contain $y_0$, the *index* $I(F, y_0)$ of $F$ *relative to* $y_0$ is defined by putting $I(F, y_0) = I(\bar{F})$, where $\bar{F}\colon S^n \to P^{n+1}$ is defined by $\bar{F}(x) = F(x) - y_0$.

The notion of homotopy of multi-valued continuous acyclic mappings was introduced in [7]. Although in [7] only the multi-valued mappings of compact spaces into compact ones were considered, condition $1^0$ of par. **2** provides that all the results of [7] remain true in the general case, if homology with compact carriers is used. In particular, if $F, G\colon X \to Y$ are two multi-valued continuous acyclic mappings which are acyclically homotopic, then $F_* = G_*$. It follows that, *if* $F, G\colon S^n \to P^{n+1}$ *are two non-vanishing multi-valued continuous acyclic mappings which are acyclically homotopic* (*in* $P^{n+1}$), *then* $I(F) = I(G)$. If the single-valued mapping $f\colon S^n \to P^{n+1}$ is a continuous selector of $F\colon S^n \to P^{n+1}$, then $I(F) = I(f)$ and $I(f)$ is the usual Kronecker index of $f$.

Let $Q^{n+1}$ be the closed spherical region in $E^{n+1}$ bounded by $S^n$.

PROPOSITION. *Let* $G\colon Q^{n+1} \to E^{n+1}$ *be a multi-valued continuous acyclic mapping which does not vanish on* $S^n$. *If* $I(G|S^n) = 0$, *then* $G$ *vanishes in some interior point of* $Q^{n+1}$, *i. e. there exists a point* $x_0 \in Q^{n+1}$ *such that* $0 \in G(x_0)$.

Proof. Let $F = G|S^n$. If $G$ does not vanish on $Q^{n+1}$, then it may be regarded as a mapping $Q^{n+1} \to P^{n+1}$. Let $W$ be the graph of $G$ and $V$ — the graph of $F$. Consider the following commutative diagram:




where $i, j$ are injections, and $p_1, q_1$ and $p_2, q_2$ are the natural projections of the graphs $W$ and $V$, respectively. By the Vietoris Theorem, $p_{1*}$ and $p_{2*}$ are isomorphisms. Since $i_*$ is zero, it follows that $F_* = q_{2*}p_{2*}^{-1}$ is zero. Consequently, $I(F) = 0$.

**6.** The multi-valued mapping $\Phi\colon X \to X$ is called a multi-valued *involution*, if the graph of it is symmetric relative to the diagonal of $X \times X$, i. e. if $y \in \Phi(x)$ implies $x \in \Phi(y)$, for every $(x, y) \in X \times X$. The following theorem is a slight modification of Theorem 1 of [6]; it will be needed for proving Theorem 2 of par. **7** which is the main theorem of this note:

THEOREM 1. *Let* $X$ *be an* $n$*-acyclic space and let* $\Phi\colon X \to X$ *be a multi-valued continuous acyclic involution. Let* $F\colon X \to E^{n+1}$ *be a multi-valued continuous acyclic mapping satisfying the following condition: no radius in* $E^{n+1}$ *beginning at the origin intersects both the sets* $F(x)$ *and* $F(y)$, *for any* $x, y$ *such that* $y \in \Phi(x)$. *Then* $F$ *induces a non-zero homomorphism* $F\colon H_n(X) \to H_n(P^{n+1})$.

Proof. First, the homomorphism $F$ is defined, since $F(X)$ cannot contain the origin. Let $W$ be the graph of $F$ and let

$$Z = \{(x, y, u, v) : u \in F(x),\ v \in F(y),\ v \in \Phi(u)\} \subset X \times X \times P^{n+1} \times P^{n+1}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{f} & P^{n+1} & \xrightarrow{r} & S^n \\ \downarrow{\scriptstyle s} & & \uparrow{\scriptstyle q} & & \\ X & \xleftarrow{p} & W & & \end{array}$$

where $p$ and $q$ are the natural projections defined by (1) in par. 4, and the mappings $s$, $f$ and $r$ are defined by formulae:

$$s(x, y, u, v) = x,$$

$$r(u) = \frac{u}{|u|},$$

$$f(x, y, u, v) = u - v.$$

Let us observe that $f$ is well defined, for $f(x, y, u, v) = 0$ implies that the points $u \in F(x)$ and $v \in F(y)$ lie on the same radius beginning at the origin, which contradicts the assumption of the Theorem 1. Evidently, $r_*$ is an isomorphism and, by the Vietoris Theorem, $p_*$ is an isomorphism. The mapping $s$ can be factored into three mappings: $(x, y, u, v) \to (x, y, u) \to (x, y) \to x$, each of which has as counter-images, the sets homeomorphic to $F(y)$, $F(x)$, and $\Phi(x)$, respectively. By the Vietoris Theorem, each of these three mappings induces a homology isomorphism; hence, $s$ is an isomorphism. It follows that $Z$ is $n$-acyclic.

Let $\varphi : Z \to Z$ be the involution defined by

$$\varphi(x, y, u, v) = (y, x, v, u).$$

Then the mapping $g = rf : Z \to S^n$ satisfies the condition

$$g\varphi(z) = -z$$

for every $z \in Z$. By theorem 6 of [5] it follows that the homomorphism $g_* : H_n(Z) \to H_n(S^n)$ is non-zero. It follows that $q_* \neq 0$, and then $F_* = q_* p_*^{-1} \neq 0$.

Corollary 1. *Let $F : S^n \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that*

(2) *no radius beginning at the origin intersects both the sets $F(x)$ and $F(-x)$, for any $x \in S^n$.*

*Then $I(F) \neq 0$.*

Corollary 2. *Let $G : Q^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that the mapping $F = G|S^n$ satisfies condition (2). Then $G$ vanishes in some interior point of $Q^{n+1}$.*

**7.** Let $A$ be a subset of $E^{n+1}$. The point $a \in A$ is an interior point of $A$ if and only if the kernel of the homomorphism $H_n(A\setminus(a)) \to H_n(A)$ induced by injection is non-zero. The following lemma characterizes the interior points by means of the notion of Index:

LEMMA 1. *The point $a \in A$ is an interior point of $A$ if and only if there exists a multi-valued continuous acyclic mapping $G: Q^{n+1} \to A$ such that $G(S^n) \subset A\setminus(a)$ and $I(G|S^n, a) \neq 0$.*

Proof. The necessity is obvious. To prove that the condition is sufficient, let us put $F = G|S^n$ and observe that the kernel of $H_n(A\setminus(a)) \to H_n(A)$ contains $F_*(H_n(S^n))$ which is non-zero by the assumption of Lemma.

The multi-valued mapping $F: X \to Y$ is said to be an *$\varepsilon$-mapping* ($\varepsilon > 0$) if $F(x_1) \cap F(x_2) \neq 0$ implies $\varrho(x_1, x_2) < \varepsilon$, for every $x_1, x_2 \in X$. The following theorem is a generalization of Borsuk's classical theorem on $\varepsilon$-mappings of the Euclidean space [2]:

THEOREM 2. *Let $\varepsilon$ be a positive number and let $G: E^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic $\varepsilon$-mapping. Then $G(E^{n+1})$ is an open subset of $E^{n+1}$.*

The proof is based on the following

LEMMA 2. *Let $G: Q^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic 1-mapping. Let $F = G|S^n$ and let $y_0 \in G(0)$. Then $I(F, y_0) \neq 0$.*

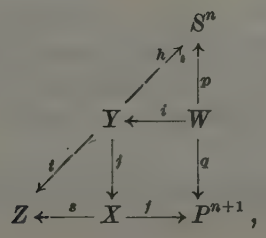
Proof. Let $\bar{G}(x) = G(x) - y_0$; then $0 \in \bar{G}(0)$. Let $W$ be the graph of $\bar{F} = \bar{G}|S^n$, i. e.

$$W = \{(x, u) : x \in S^n;\ u \in \bar{G}(x)\}.$$

Let

$$Z = \{(x, y) : x, y \in Q^{n+1};\ \varrho(x, y) = 1\},$$
$$X = \{(x, y, u, v) : (x, y) \in Z;\ u \in \bar{G}(x);\ v \in \bar{G}(y)\},$$
$$Y = \{(x, y, u, v) \in X : y = 0\}.$$

Let us observe that if $(x, u) \in W$, then $(x, 0, u, 0) \in Y$. Therefore, $(x, u) \to (x, 0, u, 0)$ defines a mapping $i: W \to Y$. Consider the following commutative diagram:



In this diagram, $p$ and $q$ are the natural projections, defined as in par. **4**, $j$ is the injection and $s$, $t$, $h$ and $f$ are defined by:

$$s(x, y, u, v) = (x, y),$$
$$t(x, 0, u, v) = (x, 0),$$
$$h(x, 0, u, v) = x,$$
$$f(x, y, u, v) = u - v.$$

Notice that $t$ is well defined, for $(x, 0, u, v) \in Y$ implies $\varrho(x, 0) = 1$; and so is $f$, for $f(x, y, u, v) = 0$ implies $u = v \in \bar{G}(x) \cap \bar{G}(y)$ which means that $G(x) \cap G(y) \neq 0$ and $\varrho(x, y) = 1$.

Since $F_* = q_* p_*^{-1}$, it should be proved that $q_* : H_n(W) \to H_n(P^{n+1})$ is non-zero, for $p_*$ is an isomorphism. The mapping $s$ can be factored into two mappings $(x, y, u, v) \to (x, y, u) \to (x, y)$ and each of them has acyclic counter-images. It follows from the Vietoris Theorem that $s_*$ is an isomorphism. In the same way we prove that $t_*$ and $h_*$ are isomorphisms. It follows from the commutativity that $i_*$ and $j_*$ are isomorphisms. Since $S^n$ is $n$-acyclic, therefore, $X$ is $n$-acyclic.

Let $\varphi : X \to X$ be the (single-valued) involution defined by

$$\varphi(x, y, u, v) = (y, x, v, u).$$

The mapping $f$ satisfies the condition

$$f\varphi(x, y, u, v) = -f(x, y, u, v).$$

Therefore, the assumptions of Theorem 1 of par. **6** are fulfilled, when $\varphi$ is taken for $\Phi$ and $f$ for $F$. It follows that $f_* \neq 0$ and then $q_* \neq 0$.

Proof of Theorem 2. Let $y_0 \in G(E^{n+1})$ and let $x_0 \in E^{n+1}$ be such that $y_0 \in G(x_0)$. Let $Q_\varepsilon^{n+1}$ be the closed spherical region with the centre $x_0$ and the radius $\varepsilon$ and $S_\varepsilon^n = \dot{Q}_\varepsilon^{n+1}$. Let $G_\varepsilon = G|Q_\varepsilon^{n+1}$, $F_\varepsilon = G_\varepsilon|S_\varepsilon^n$. Applying Lemma 2 to $Q_\varepsilon^{n+1}$, $S_\varepsilon^n$, $G_\varepsilon$, $F_\varepsilon$ and $\varepsilon$, instead of to $Q^{n+1}$, $S^n$, $G$, $F$ and 1 and then applying Lemma 1 we infer that $y_0$ is an interior point of $G(E^{n+1})$.

THEOREM 3 (ON THE INVARIANCE OF DOMAIN). *Let $U$ be an open subset of $E^{n+1}$ and let $G : U \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that the sets $F(x_1)$ and $F(x_2)$ are disjoint, for every $x_1 \neq x_2$. Then $G(U)$ is open in $E^{n+1}$.*

Proof. It follows that $G$ is an $\varepsilon$-mapping, for every $\varepsilon > 0$; let $y_0 \in G(x_0)$, where $x_0 \in U$. Let $Q_\varepsilon^{n+1}$ be a closed spherical region, with the centre $x_0$ and the radius $\varepsilon > 0$, contained in $U$. Since $G|Q_\varepsilon^{n+1}$ is an $\varepsilon$-mapping, it follows by Lemma 2 and Lemma 1 that $y_0$ is an interior point of $G(U)$.

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# The Numerical Values of the Two-centre Integrals for 3d Electrons in Transition Metals

by

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*Presented by L. INFELD on March 10, 1959*

The numerical results of tight-binding calculations depend ultimately on the values of the constants in the matrix components of energy. The necessary energy integrals are the three-centre integrals but the two-centre approximation replaces them by the resonance integrals frequently encountered in molecular studies. It can be hoped that, in a few cases of physical interest, calculations based on the tight-binding method may be of some use. The investigation on the 3d electron band in the transition metals might be one such instance.

Fletcher and Wohlfarth [1], [2] suggested the calculation of energy integrals for tight-binding in the two-centre approximation and based their calculation of the 3d electron band in nickel on computed values of the two-centre integrals.

Slater and Koster [3] developed in a systematic way the two-centre approximation in the tight-binding scheme for cubic lattices. M. Miąsek [4], [5] extended the tables of Slater and Koster to the close-packed hexagonal structure.

The calculation of the two-centre energy integrals for 3d electrons has now been undertaken, for iron, cobalt, nickel, copper and zinc. The radial part of the atomic 3d wave functions have been approximated by the analytical expressions of Löwdin type [6], actually with two exponentials only [7], [8]:

$$P_{A(r)} = r^3(A_1 e^{-\alpha_1 r} + A_2 e^{-\alpha_2 r}) . \tag{1}$$

The atomic wave function for iron and cobalt was approximated to the numerical $P(3d\,1; r)$ function calculated with exchange, for iron, by J. H. Wood [9]. The atomic wave function for nickel, copper and zinc was approximated to the numerical 3d function calculated with exchange, for Cu*, by Hartree [10].

In the present calculations the exact formulas for the two-centre energy integrals have been employed. These formulas are published in full in [11], [12] *). The approximations used by Fletcher [2] have been found, in the course of the present calculations, to be very good indeed.

Let us write down the values of the integrals we have obtained. In the case of each metal the integrals will be given here for the nearest neighbour only. In the first place we write the $(dd\sigma)$ integral, then $(dd\pi)$, and then the $(dd\delta)$ integral, in the notation of the tables of Slater and Koster [3]. All the energy integrals are given in electron-volts. In the second line are given the corresponding overlap integrals between the wave functions $3d\sigma$, $3d\pi$ and $3d\delta$, [13], [14]. Our overlap integrals are reduced to unity for zero distance, as we use normalized wave functions.

The numerical values are:

for the body-centered cubic iron, with the atomic number $Z = 26$ and the lattice constant $a = 2.86645 \cdot 10^{-8}$ cm

| | | | |
|---|---|---|---|
| (2) | −0.4118 | 0.2214 | −0.0333 |
| (2′) | 0.0313 | −0.0195 | 0.0033 |

for the close-packed hexagonal cobalt, with $Z = 27$, $a = 2.5074 \cdot 10^{-8}$ cm

| | | | |
|---|---|---|---|
| (3) | −0.3945 | 0.2084 | −0.0309 |
| (3′) | 0.0302 | −0.0185 | 0.0031 |

for the face-centered cubic nickel, with $Z = 28$, $a = 3.52394 \cdot 10^{-8}$ cm

| | | | |
|---|---|---|---|
| (4) | −0.3787 | 0.2112 | −0.0330. |
| (4′) | 0.0265 | 0.0177 | 0.0032. |

It is definitely the tail of the atomic wave function which mostly determines the values of the two-centre integrals [15]. The choice of the smallest exponent is very important and the fact that, in fitting the numerical wave function, this exponent can be taken in slightly different ways, constitutes the main uncertainity in our computed values.

In order to see how much the values of the energy integrals vary in dependence on the least-exponent term we quote here the values of the energy integrals for the nearest neighbour in iron, obtained with the different choices of the least-exponent term. It must be said that all these analytical wave functions fit the numerical normalized wave function $P(3d\,1; r)$ of J. H. Wood reasonably well. The above values (2)

*) It should be noted here that, in references [11], [12], the proper sign of the $(dd\pi)$ integral is obtained, if the two coordinate systems, $xyz$ and $x'y'z'$, used there will both be made right-handed. Also an error in the fromula (21) of [12] should be corrected on the present occasion. The term $270/a^5$ should be replaced by $2700/a^5$. The fact that, in the present calculations carried out on an electronic computer, the numerical values of references [2] and [8] have been satisfactorily confirmed, gives confidence in the present, as well as the previous, numerical work.

have been computed with the analytical wave function (1) having $a_1 = 2.1$, $a_2 = 5.1$, $A_1 = 2.815$, $A_2 = 81.97726$.

For the wave function with $a_1 = 2.2$, $a_2 = 5.1$, $A_1 = 3.459$, $A_2 = 77.80095$, the values of the energy integrals are

(5) $-0.3708$ $0.1859$ $-0.0263$

For the wave function with $a_1 = 2.2$, $a_2 = 5.1$, $A_1 = 3.19$, $A_2 = 82.55929$, the values of the energy integrals are

(6) $-0.3308$ $0.1638$ $-0.0230$

With $a_1 = 2.25$, $a_2 = 5.1$, $A_1 = 3.597$, $A_2 = 79.44393$, they are

(7) $-0.3210$ $0.1543$ $-0.0211$

Further, with $a_1 = 2.3$, $a_2 = 5$, $A_1 = 4.2$, $A_2 = 68.22085$, they are

(8) $-0.3292$ $0.1543$ $-0.0206$

Thus, the dependence of the numerical values of the two-centre integrals on the least-exponent term is very pronounced.

The numerical values for the further neighbours and for other metals will be published together with the details of the calculations on another occasion.

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# Electron Polarization in First Forbidden Beta Transitions with Spin Change $\Delta J \pm 1$

by

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## Introduction

In this paper the electron polarization is investigated in first forbidden beta transitions with spin change $\Delta J = \pm 1$. For this case no formulae have been given hitherto, which could be used in the experiments connecting the measurement of electron polarization with the angular $e-\nu$ correlation. Only formulae for the $e-\nu$ correlation are available [1], as well as some leading terms for Coulomb transitions [2].

The general expressions in the present paper are derived for the universal Fermi interaction with two complex coupling constants $C_V$ and $C_A$ (this reflects the fact that the mixing ratio of the $V$ and $A$ couplings may differ from unity and no strong support exists for time reversal invariance). In agreement with the present knowledge on $\beta$-interaction it is assumed that the parity non-conserving coupling constants $C_V a_V$ and $C_A a_A$ (in the notation of [3]) are equal to $C_V$ and $C_A$, respectively. A generalization of the formulae for the case of $a_i \neq 1$ is simple and may be performed in the usual manner.

## Polarization formulae

The calculations have been performed by the same trace method as in the preceding papers by this author. The notation and the coordinate system are the same as in [3].

The expressions for the longitudinal polarization $P_x$ of the electron and the two transverse polarization components $P_y$ and $P_z$ are the following:

$$P_x = \frac{H_2}{H_1} \qquad P_y = \frac{H_3}{H_1}\sin\vartheta \qquad P_z = \frac{H_4}{H_1}\sin\vartheta\,,$$

where the functions $H_i$ $(i = 1, 2, 3, 4)$ are given below:

$$
\begin{aligned}
H_i = {} & |C_A|^2 \left| \int \vec{\sigma} \times \vec{r} \right|^2 A_i + |C_V|^2 \left\{ \left| \int \vec{a} \right|^2 \cdot V_{i,1} + \left| \int \vec{r} \right|^2 V_{i,2} \right. \\
& \left. + 2\,\mathrm{Re}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{r} \right) \right] V_{i,3} + 2\,\mathrm{Im}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{r} \right) \right] \cdot V_{i,4} \right\} \\
& + 2\,\mathrm{Re}(C_A C_V^*) \left\{ 2\,\mathrm{Re}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,1} + 2\,\mathrm{Im}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,2} \right. \\
& \left. + 2\,\mathrm{Re}\left[ \left( \int \vec{r} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,3} + 2\,\mathrm{Im}\left[ \left( \int \vec{r} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,4} \right\} \\
& + 2\,\mathrm{Im}(C_A C_V^*) \left\{ 2\,\mathrm{Re}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,2} \right. \\
& - 2\,\mathrm{Im}\left[ \left( \int \vec{a} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,1} + 2\,\mathrm{Re}\left[ \left( \int \vec{r} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,4} \\
& \left. - 2\,\mathrm{Im}\left[ \left( \int \vec{r} \right)^* \left( \int \vec{\sigma} \times \vec{r} \right) \right] \cdot I_{i,3} \right\} .
\end{aligned}
$$

The coefficients $A_i$, $V_{ik}$ and $I_{ik}$ are listed below:

$$
\begin{aligned}
A_1 = {} & \tfrac{1}{3} q^2 L_0 + \tfrac{4}{3} q N_0 + 2 M_0 + L_1 + q \hat{K}^+_{c1,-2} (\tfrac{1}{3} - \cos^2\vartheta) \\
& + [ - \tfrac{1}{3} q^2 L_0' - \tfrac{4}{3} q N_0' + \tfrac{2}{3} q K^+_{c1,2} + \tfrac{8}{3} \hat{K}^-_{c-1,-2} - \tfrac{2}{3} M_0' + \tfrac{1}{3} L_1' ] \cos\vartheta \\
A_2 = {} & - \tfrac{1}{3} q^2 L_0' - \tfrac{4}{3} q N_0' - L_1' - 2 M_0' - q K^+_{c1,2} (\tfrac{1}{3} - \cos^2\vartheta) \\
& + [ \tfrac{1}{3} q^2 L_0 + \tfrac{4}{3} q N_0 - \tfrac{2}{3} q K^+_{c1,-2} - \tfrac{8}{3} K^-_{c-1,2} + \tfrac{1}{3} L_1 + \tfrac{2}{3} M_0 ] \cos\vartheta \\
A_3 = {} & \tfrac{1}{3} q^2 P_0 + \tfrac{4}{3} q R_0 - \tfrac{1}{3} q \hat{K}^-_{c1,-2} - \tfrac{2}{3} Q_0 - \tfrac{2}{3} P_1 + \tfrac{4}{3} K^+_{c-1,2} - q K^-_{c1,2} \cdot \cos\vartheta \\
A_4 = {} & \tfrac{4}{3} q N_0'' + \tfrac{1}{3} q K^-_{s1,2} - \tfrac{2}{3} P_1' - \tfrac{2}{3} Q_0' - \tfrac{4}{3} \hat{K}^+_{s-1,-2} + q \cdot \hat{K}^-_{s1,-2} \cdot \cos\vartheta \\
V_{11} = {} & 2 (L_0 - \tfrac{1}{3} L_0' \cos\vartheta) \\
V_{21} = {} & 2 (- L_0' + \tfrac{1}{3} L_0 \cos\vartheta) \\
V_{31} = {} & \tfrac{2}{3} P_0 \\
V_{41} = {} & \tfrac{2}{3} P_0' \\
V_{12} = {} & \tfrac{2}{3} q^2 L_0 - \tfrac{4}{3} q N_0 + 4 L_1 + 2 M_0 - 4 q \hat{K}^+_{c1,-2} (\tfrac{1}{3} - \cos^2\vartheta) \\
& + [ \tfrac{2}{3} q^2 L_0' - \tfrac{4}{3} q N_0' + \tfrac{8}{3} q K^+_{c1,2} - \tfrac{16}{3} \hat{K}^-_{c-1,-2} + \tfrac{4}{3} L_1' - \tfrac{2}{3} M_0' ] \cos\vartheta \\
V_{22} = {} & - \tfrac{2}{3} q^2 L_0' + \tfrac{4}{3} q N_0' - 8 L_1' - 2 M_0' + 4 q K^+_{c1,2} (\tfrac{1}{3} - \cos^2\vartheta) + [ - \tfrac{2}{3} q^2 L_0 + \tfrac{4}{3} q N_0 \\
& - \tfrac{8}{3} q \hat{K}^+_{c1,-2} - \tfrac{4}{3} L_1 + \tfrac{2}{3} M_0 + \tfrac{16}{3} K^-_{c-1,2} ] \cdot \cos\vartheta \\
V_{32} = {} & - \tfrac{2}{3} q^2 P_0 + \tfrac{4}{3} q R_0 - \tfrac{4}{3} q \hat{K}^-_{c1,-2} - \tfrac{8}{3} K^+_{c-1,2} - \tfrac{2}{3} Q_0 - \tfrac{8}{3} P_1 + 4 K^-_{c1,2} \cdot \cos\vartheta \\
V_{42} = {} & - \tfrac{2}{3} q^2 P_0' + \tfrac{4}{3} q N_0'' + \tfrac{4}{3} q K^-_{s1,2} - \tfrac{8}{3} P_1' - \tfrac{2}{3} Q_0' + \tfrac{8}{3} \hat{K}^+_{s-1,-2} - 4 q \hat{K}^-_{s1,-2} \cdot \cos\vartheta \\
V_{13} = {} & - 4 \hat{K}^+_{s1,-2} + [ \tfrac{2}{3} R_0' + \tfrac{4}{3} K^+_{s1,2} ] \cdot \cos\vartheta \\
V_{23} = {} & 2 R_0' + \tfrac{8}{3} \hat{K}^+_{s1,-2} \cos\vartheta \\
V_{33} = {} & \tfrac{4}{3} \hat{K}^-_{s1,-2} \\
V_{43} = {} & \tfrac{4}{3} K^-_{c1,2} - \tfrac{2}{3} R_0''
\end{aligned}
$$

$$V_{14} = \tfrac{2}{3} qL_0 - 2N_0 + [\tfrac{2}{3} qL_0' + \tfrac{2}{3} N_0' + \tfrac{8}{3} K_{c1,2}^{+}] \cdot \cos\vartheta$$

$$V_{24} = -\tfrac{2}{3} qL_0' + 2N_0' + [-\tfrac{2}{3} qL_0 - \tfrac{2}{3} N_0 - \tfrac{8}{3} \hat{K}_{c1,-2}^{+}] \cdot \cos\vartheta$$

$$V_{34} = -\tfrac{2}{3} qP_0 - \tfrac{2}{3} R_0 - \tfrac{4}{3} \hat{K}_{c1,-2}^{-}$$

$$V_{44} = -\tfrac{2}{3} qP_0' + \tfrac{4}{3} K_{s1,2}^{-} - \tfrac{2}{3} N_0''$$

$$I_{11} = \tfrac{1}{3} qL_0 + N_0 + [-\tfrac{1}{3} qL_0' - \tfrac{1}{3} N_0' + \tfrac{2}{3} K_{c1,2}^{+}] \cdot \cos\vartheta$$

$$I_{21} = -\tfrac{1}{3} qL_0' - N_0' + [\tfrac{1}{3} qL_0 + \tfrac{1}{3} N_0 - \tfrac{2}{3} \hat{K}_{c1,-2}^{+}] \cdot \cos\vartheta$$

$$I_{31} = \tfrac{1}{3} qP_0 + \tfrac{1}{3} R_0 - \tfrac{1}{3} \hat{K}_{c1,-2}^{-}$$

$$I_{41} = -\tfrac{1}{3} qP_0' - \tfrac{1}{3} K_{s1,2}^{-} - \tfrac{1}{3} N_0''$$

$$I_{12} = (\tfrac{1}{3} R_0' - \tfrac{2}{3} K_{s1,2}^{+}) \cos\vartheta$$

$$I_{22} = R_0' - \tfrac{5}{6} \hat{K}_{s1,-2}^{+} \cdot \cos\vartheta$$

$$I_{32} = -\tfrac{1}{3} \hat{K}_{s1,-2}^{-}$$

$$I_{42} = -\tfrac{1}{3} R_0'' - \tfrac{1}{3} K_{c1,2}^{-}$$

$$I_{13} = -2\hat{K}_{s-1,-2}^{-} - q\hat{K}_{s1,-2}^{+}(\tfrac{1}{3} - \cos^2\vartheta)$$

$$I_{23} = \tfrac{2}{3} qR_0' - \hat{K}_{s-1,-2}^{-} + qK_{s1,2}^{+}(\tfrac{1}{3} - \cos^2\vartheta) + [-\tfrac{1}{3} q\hat{K}_{s1,-2}^{+} + 2K_{s-1,2}^{-}] \cos\vartheta$$

$$I_{33} = -K_{s-1,2}^{+} + q \cdot K_{s1,2}^{-} \cdot \cos\vartheta$$

$$I_{43} = \hat{K}_{c-1,-2}^{+} - q\hat{K}_{c1,-2}^{-} \cdot \cos\vartheta$$

$$I_{14} = -L_1 + M_0 + [-\tfrac{2}{3} qN_0' - \tfrac{2}{3} qK_{c1,2}^{+} - \tfrac{2}{3} \hat{K}_{c-1,-2}^{-} - \tfrac{1}{3} L_1' - \tfrac{1}{3} M_0'] \cdot \cos\vartheta$$

$$I_{24} = -M_0' - \hat{K}_{c-1,-2}^{-} + [\tfrac{1}{3} qN_0 + \tfrac{1}{3} q\hat{K}_{c1,-2}^{+} + \tfrac{1}{3} M_0 + \tfrac{1}{3} L_1 + \tfrac{2}{3} K_{c-1,2}^{-}] \cdot \cos\vartheta$$

$$I_{34} = \tfrac{1}{6} qR_0 + \tfrac{5}{6} q\hat{K}_{c1,-2}^{-} + \tfrac{2}{3} P_1 - \tfrac{1}{3} Q_0 - \tfrac{1}{3} K_{c-1,2}^{+}$$

$$I_{44} = -\tfrac{2}{3} qN_0'' + \tfrac{1}{3} qK_{s1,2}^{-} + \tfrac{2}{3} P_1' - \tfrac{1}{3} Q_0' + \tfrac{1}{3} \hat{K}_{s-1,-2}^{+}.$$

In the formulae we have used the standard beta decay functions $L_i$ etc., as well as some new combinations of electron radial wave functions. These new combinations are defined in the paper of Curtis and Lewis [4], in the preceding paper of the present author [3] and in the Appendix to this paper. Limiting values for $\frac{\alpha Z}{p} \ll 1$, and $p\varrho \ll 1$ (where $p$ is the electron momentum and $\varrho$ – the nuclear radius) have also been given. It is thus easy to write the asymptotic form of the polarization-correlation formulae, which was omitted because of lack of space.

So far our formulae are valid for $\Delta J = \pm 1$ transitions, with the exception of $0 \leftrightarrow 1$ transitions. If the latter transitions have to be treated, terms containing the matrix element $\sum |B_{ij}|^2$ must be added to the functions $H_i$. These terms have the same forms as for first forbidden unique transitions. They have been calculated by the author in [3] and can be taken without any alteration from that paper ($|C_A|^2 \cdot \sum |B_{ij}|^2$ multiplied by suitable coefficients $A_i$ of the formulae (3)).

In our calculations no special assumptions concerning the relative phases of nuclear matrix elements have been made. If time reversal invariance is assumed for nuclear forces, the interference terms:

$$2\,\mathrm{Re}\left[\left(\int \vec{\alpha}\right)^{*}\left(\int \vec{r}\right)\right], \quad 2\,\mathrm{Im}\left[\left(\int \vec{\alpha}\right)^{*}\left(\int \vec{\sigma}\times\vec{r}\right)\right] \quad \text{and} \quad 2\,\mathrm{Re}\left[\left(\int \vec{r}\right)^{*}\left(\int \vec{\sigma}\times\vec{r}\right)\right]$$

have to be put equal to zero.

A detailed account on first forbidden beta transitions will be published elsewhere.

The author wishes to express his thanks to Professor J. Werle for helpful advice and suggestions.

## Appendix

In paper [3] some new $\beta$-decay functions have been introduced The $K_s$ – functions can be defined generally for $|i| < |k|$:

$$K^{\pm}_{si,k} = [2p^2F_0(Z,E)]^{-1}\varrho^{-l(-i)-l(-k)}\cdot[f_i f_k \sin(\Delta_i-\Delta_k) \pm g_{-i}\,g_{-k}\sin(\Delta_{-i}-\Delta_{-k})],$$

$$\hat{K}^{\pm}_{si,k} = [2p^2F_0(Z,E)]^{-1}\varrho^{-l(-i)-l(k)}\cdot[f_i g_k \sin(\Delta_i-\Delta_k) \pm f_{-i}g_{-k}\sin(\Delta_{-i}-\Delta_{-k})].$$

The functions $K^{\pm}_{ci,k}$ and $\hat{K}^{\pm}_{ci,k}$ may be obtained by changing the sinus functions in the above definitions into cosinus functions. In these definitions, $F_0(Z, E)$ is the Fermi function, and the functions $f_k$ and $g_k$ are normalized to one particle per unit sphere. $l(k)$ is given by

$$l(k) = |k| + \tfrac{1}{2}[\mathrm{Sgn}\,k - 1].$$

The limiting values of some $K$-functions for $\dfrac{aZ}{p} \ll 1$ and $p\varrho \ll 1$ $\left(\text{but } \dfrac{\xi}{p} \sim 1\right)$ have been given in [3]. Here $\varrho$ denotes the nuclear radius and $\xi = \dfrac{aZ}{2\varrho}$. In this paper there appear also other $K$-functions, which have the following limiting values:

$$K^{+}_{c-1,2} \to \frac{1}{9}\frac{p^2}{E} \qquad \hat{K}^{+}_{c-1,-2} \to -\frac{1}{3}\frac{p\xi}{E}$$

$$K^{-}_{c-1,2} \to -\frac{p^2}{3}\left(\frac{1}{3}+\frac{\xi}{E}\right) \qquad \hat{K}^{-}_{c-1,-2} \to -\left(\frac{1}{3}p\xi + \frac{1}{9}\frac{p^2}{E}\right)$$

$$K^{+}_{s-1,2} \to -\frac{aZ}{4E}p\xi \qquad \hat{K}^{+}_{s-1,-2} \to -aZ\left(\frac{1}{3}\xi + \frac{1}{36}\frac{p^2}{E}\right)$$

$$K^{-}_{s-1,2} \to -\frac{aZp}{4}\left(\xi + \frac{1}{3}\frac{p^2}{E}\right) \qquad \hat{K}^{-}_{s-1,-2} \to -aZ\left(\frac{1}{12}p^2 + \frac{p^2\xi}{4E} + \frac{\xi}{3E}\right).$$

Three other combinations of Coulomb radial wave functions are also introduced and their limiting values are given,

$$Q_0' = [2p^2F_0(Z, E)^{-1} \cdot \varrho^{-2} \cdot 2f_{-1}g_1 \sin(\Delta_1 - \Delta_{-1}) \to \alpha Z \left(\frac{\xi^2}{E} + \frac{2}{3}\xi + \frac{p^2}{9E}\right),$$

$$N_0'' = [2p^2F_0(Z, E)]^{-1} \cdot \varrho^{-1}(f_1f_{-1} - g_1g_{-1}) \sin(\Delta_1 - \Delta_{-1}) \to \frac{\alpha Z}{E}(\xi + \tfrac{1}{3}E),$$

$$R_0'' = [2p^2F_0(Z, E)]^{-1} \cdot \varrho^{-1}(f_1f_{-1} + g_1g_{-1}) \cdot \cos(\Delta_1 - \Delta_{-1}) \to \tfrac{1}{3}\frac{p}{E}.$$

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*EXPERIMENTAL PHYSICS*

# Emission Anisotropy of Photoluminescence of Acridine Yellow in Gelatine in the Pre-excited State

by

M. FRĄCKOWIAK, J. GRZYWACZ and J. HELDT

*Presented by A. JABŁOŃSKI on March 10, 1959*

Some organophosphors change their properties as a result of pre-excitation or excitation [1], [3]. An examination of the decay curves of phosphorescence and of the photoluminescence and absorption spectra of acridine yellow in gelatine [4], [5] leads to the conclusion that intense pre-excitation gives rise to deformed luminescent centres possessing systems of electronic terms differing from those of the undeformed centres. If the photoluminescence band observed is a superposition of bands connected with transitions between various electronic terms, emission anisotropy will, in general, be different for various wave-lengths within the "composed" band. If the ratio of the intensities of the "component" bands varies during pre-excitation, emission anisotropy becomes a function of the time of pre-excitation $\tau$.

The aim of the present paper is to investigate the effect of pre-excitation on the emission anisotropy of photoluminescence of acridine yellow in gelatine. The effect of the duration of pre-excitation and of the mutual orientation of the electric vectors of the plane polarized pre-exciting and exciting light was investigated at room temperature. Measurements were carried out by the visual method, by means of an Arago compensator, Savart plate and Nicol prism, of the total light emitted in the photoluminescence band and of the emission in various spectral regions within the band (the width of each region was less than 50 Å). Fig. 1 shows the diagram of the device used for investigating the variations of the emission anisotropy as a function of the duration of pre-excitation. Depolarization produced by the geometry of the system has not been accounted for. The phosphor was subjected to continuous pre-excitation for periods of 0, 5, 10, 15 and 20 minutes with both natural and plane polarized white light. Excitation of the phosphor was carried out by means of plane polarized light of wave-lenghts belonging to the spectral region of the absorption band. When measuring emission aniso-

tropy for various spectral regions within the emission band, the interference fringes were observed through the monochromator $M$ by means of a magnifying glass $L$. When the emission anisotropy of the total band was measured, the monochromator was removed and the interference fringes were observed directly through the Nicol $N_2$ applying crossed filters. The time interval between the cessation of pre-excitation and the beginning of the measurements was 30 sec. In each case of pre-excitation, the time of measurement was 4 min. The time between two consecutive periods of pre-excitation was 3 hours. It was found that the phosphor returns to its initial non-pre-excited state, if kept in dark during this period. Prior to pre-excitation and in the intervals between

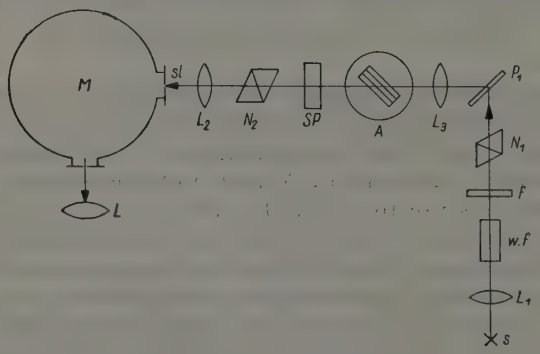


Fig. 1. Diagram of device for measuring variations of emission anisotropy. $S$ — light source, $L_1, L_2, L_3$ — lenses, $w.f.$ — water filter, $f$ — optical filter, $N_1, N_2$ — Nicol prisms, $P$ — phosphor, $A$ — Arago compensator, $M$ — Zeiss monochromator, $SP$ — Savart plate, $L$ — magnifying glass, $Sl$ — monochromator slit

pre-excitation and excitation the temperature of the phosphor was measured with an accuracy of $\pm 0.1°C$. No variations in temperature were found to arise from either pre-excitation or excitation. Emission anisotropy was measured jointly for fluorescence and phosphorescence light because of the necessity of working with a sufficient intensity of photoluminescence.

The results of measurements of emission anisotropy for the total luminescence band are given in Table I and shown in Fig. 2. It is easily seen therefrom that emission anisotropy does not depend on the mutual orientation of the electric vectors of the plane polarized pre-exciting and exciting light. It depends, however, on the duration of pre-excitation. As the latter increases, emission anisotropy diminishes. At about 15 min. pre-excitation, a state which may be described as "saturation" is observed. Further pre-excitation results in practically no variations. The variations in emission anisotropy observed should be considered to result from variations in the ratio of the intensities of the luminescence

TABLE I

Emission anisotropy *versus* time of pre-excitation and mutual orientation of electric vectors of pre-exciting and exciting light

| $\tau$ (min.) | 0 | 5 | | | 10 | | | 15 | | | 20 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\vec{E}$pre-exc. | | ↕ | ↔ | ✣ | ↕ | ↔ | ✣ | ↕ | ↔ | ✣ | ↕ | ↔ | ✣ |
| $\vec{E}$exc. | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ | ↕ |
| $r$ | 0.166 | 0.141 | 0.141 | 0.150 | 0.135 | 0.135 | 0.135 | 0.128 | 0.131 | 0.129 | 0.127 | 0.131 | 0.129 |
| $\alpha$ | 42° 12′ | 39° 29′ | 39° 29′ | 40° 54′ | 38° 53′ | 38° 53′ | 38° 53′ | 37° 18′ | 37° 37′ | 37° 20′ | 37° 16′ | 37° 37′ | 37° 20′ |
| | | ○ | △ | | ○ | △ | | ○ | △ | | ○ | △ | |

bands of the undeformed and deformed centres, $I_1/I_2$. At longer periods of pre-excitation, the number of deformed centres grows at the cost of that of the undeformed ones. Thus, $I_2$ augments as $I_1$ diminishes. Hence and from the fact that the emission anisotropy decreases in pre-excited

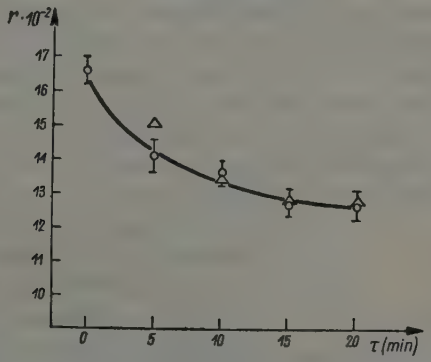


Fig. 2. Emission anisotropy as measured in total luminescence band *versus* time of pre-excitation and mutual orientation of electric vectors of plane polarized pre-exciting and exciting light

phosphors, the emission anisotropy of deformed centres may be concluded to be less than that of the undeformed ones.

By the results of [4]-[6], the reduction of the emission anisotropy of the deformed centres is due rather to the lesser value of the fundamental emission anisotropy of the deformed molecules than to a rise in the probability of migration of excitation energy in the deformed centres. The state of "saturation" arises from equilibrium between the growth of the number of deformed centres and their return to the initial (undeformed) state.

Table II and Fig. 3 show the values of the emission anisotropy of luminescence *versus* the time of pre-excitation. Each of the values of

TABLE II

Emission anisotropy of luminescence *versus* emitted wave-length and time of pre-excitation

| $\tau$ (min.) | $\lambda = 5100$ Å | | $\lambda = 5200$ Å | | $\lambda = 5400$ Å | | $\lambda = 5700$ Å | |
|---|---|---|---|---|---|---|---|---|
| | $\bar{a}$ | $r$ | $\bar{a}$ | $r$ | $\bar{a}$ | $r$ | $\bar{a}$ | $r$ |
| 0 | 45°11′ | 0.194 | 42°12′ | 0.166 | 40°20′ | 0.147 | 37°15′ | 0.126 |
| 5 | 45°00′ | 0.193 | 41°53′ | 0.162 | 39°31′ | 0.143 | 37°00′ | 0.123 |
| 10 | 43°32′ | 0.179 | 41°20′ | 0.158 | 39°12′ | 0.139 | 35°36′ | 0.113 |
| 15 | 42°00′ | 0.163 | 41°00′ | 0.154 | 38°54′ | 0.136 | 34°06′ | 0.099 |

the angle $\alpha$ of the Arago compensator given in Tables I and II represents the arithmetical mean of 10 measurements. The growth of the intensity of the luminescence band of the deformed centres may be reconstructed from the data of Fig. 3. The curve for $\lambda = 5100$ Å (Fig. 3) in the 0—5 minutes range of pre-excitation is parallel to the axis of abscissae, which points to the fact that, throughout this range, emission anisotropy is related solely to the undeformed centres. At longer times of pre-excitation, the curve falls steeply owing to expansion of the short-wave wing of the luminescence band of the deformed centres. The shape of the curves for $\lambda = 5200$ Å and $\lambda = 5400$ Å is determined by the variations of the intensity of the bands. That for $\lambda = 5700$ Å changes but insignificantly at short pre-excitation showing to be related to the emission by the undeformed centres.

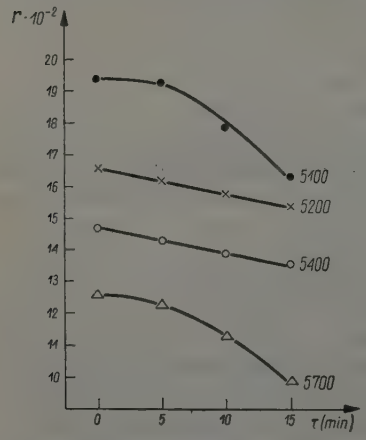


Fig. 3. Emission anisotropy *versus* time of pre-excitation, at various wave-lengths of emitted light

The difference in emission anisotropy at short pre-excitation for $\lambda = 5100$ Å and $\lambda = 5700$ Å may be due to superposition of slow fluorescence band on the phosphorescence band in the $\lambda = 5700$ Å region. The steep fall of this curve at longer periods of pre-excitation is due to expansion of the long-wave wing of the luminescence band of the deformed centres (as in the case of the $\lambda = 5100$ Å curve).

Fig. 4 shows the variation of emission anisotropy *versus* the wave-length within the luminescence band, for various periods of pre-excitation.

The shape of the various curves (including that of $\tau = 0$, because of long excitation) may be caused not only by superposition of luminescence bands of the deformed and undeformed centres and by certain effects, previously mentioned, produced by the device, but also by the complex structure of the bands [8], [10]. However, the differences of the shape of the curves as shown in Fig. 4 are determined solely by effects of superposition of the luminescence bands of the deformed centres and those of the undeformed ones.

From Figs. 3 and 4, it is possible to determine the position of the centre of "gravity" of the luminescence band of the deformed centres,

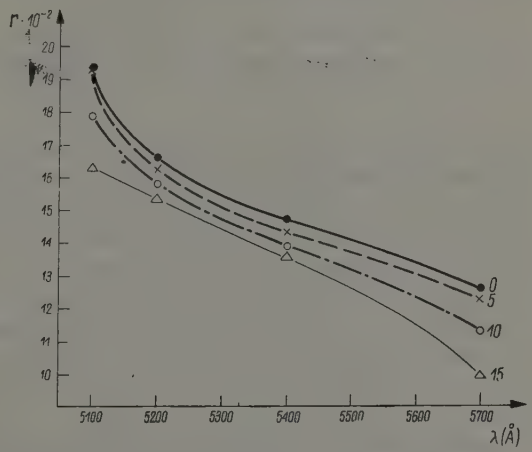


Fig. 4. Emission anisotropy *versus* wave-length of emitted light, at various times of pre-excitation

which lies between 5200 and 5500 Å, in the short-wave region of the band of emission of the undeformed centres.

From the present investigation it may be concluded that pre-excitation gives rise to deformed centres possessing their own luminescence bands. Owing to superposition of the luminescence bands of the deformed centres and those of undeformed ones, emission anisotropy varies according to the wave-length throughout a "combined" band. Owing to variations with the time of pre-excitation $\tau$ in the numbers of deformed and undeformed centres, also the anisotropy becomes a function of $\tau$.

The authors wish to express their indebtedness to Professor A. Jabłoński for his valuable advice and for his interest in the present investigation.

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*EXPERIMENTAL PHYSICS*

# Quenching of Photoluminescence of Liquid and Rigid Solutions

by

J. GŁOWACKI and R. POHOSKI

*Presented by A. JABŁOŃSKI on March 20, 1959*

**1.** It is the aim of the present paper to check experimentally the simplified version of the theory of quenching proposed by A. Jabłoński [1], who considers the centre to consist of an excited luminescent molecule enclosed in layers of the solvent containing the quenching molecules. In a subsequent paper [2], the problem of quenching is solved for the case of quenching by energy transfer between the excited and quenching molecules following Förster's assumption [3] that the quenching probability is inversely proportional to the sixth power of the distance between the quencher and the excited luminescent molecule. The following expression is obtained for the mean relative yield of quenching $\bar{\eta}_k/\eta_0$, for a centre containing $k$ quenchers within a sphere of radius $R'$:

$$\frac{\bar{\eta}_k}{\eta_0} = \frac{\bar{\eta}_k}{\bar{\eta}_{k-1}} \cdot \frac{\bar{\eta}_{k-1}}{\eta_0}, \tag{1}$$

here,

$$\frac{\bar{\eta}_k}{\bar{\eta}_{k-1}} = 1 - \frac{\operatorname{arctg}\left(\frac{R'}{\varrho_{k-1}}\right)^3}{\left(\frac{R'}{\varrho_{k-1}}\right)^3}. \tag{2}$$

If the centre contains no quenchers ($k = 0$), the relative yield $\eta_0'/\eta_0$ is given by

$$\frac{\eta_0'}{\eta_0} = \frac{1}{1 + \left(\frac{R_0}{R'}\right)^6 \cdot \nu}. \tag{3}$$

Here, the following notation has been adopted:

$R_0$ — denotes the distance between quencher and excited molecule at which the quenching probability equals that of emission,

$v = nv$, wherein $n$ is the number of quenching molecules per unit volume and $v$ is the volume of the active sphere,

$\varrho_{k-1} = R_0 \eta_{k-1}^{-6}$,

$R' = 1.327 R_0$.

The relations (1), (2) and (3) are approximate and simplified. They fail to take into consideration, among other factors, the mutual orientation of the molecules; neither do they account for Boltzmann's factor which, considering the presence of Van der Waals forces, surely affects the distribution of configuration of the centres.

**2.** For measuring the relative yield of quenching, the photoelectric method was used. Liquid solutions were investigated at a thickness of the layer amounting to 0.025 cm, whereas rigid solutions were formed into plates 0.1 cm thick.

The measurements were carried out with the following dyes: fluorescein quenched with rhodamin B in ethyl alcohol, trypaflavine quenched with rhodamin B in ethyl alcohol, and fluorescein quenched with rhodamin B in methyl polymethacrylate (plexiglass) thoroughly purified by triple distillation before polymerization until the luminescence excited therein by light from a mercury lamp transmitted through a Wood filter was hardly perceptible. Solubility of rhodamin B in methyl methacrylate was raised by an addition of ethyl alcohol (10%). Polymerization was carried out in sealed test-tubes at 60°C. The samples were cut and all the faces were subjected to polishing, the degree of accuracy being checked microscopically.

All solutions were excited through a blue filter transmitting light of wave-lengths inferior to 4900 Å. Fluorescence intensity was measured through a filter in the spectral range of 5100 to 5250 Å. The degree of polarization of luminescence light, the knowledge of which is necessary for introducing corrections for the yield in the case of a rigid solution, was measured by the visual method using a Savart plate and Arago compensator, and was found to be 24%, remaining constant throughout the entire range of the solutions investigated. It is worth noting that already Kawski [4] observed the absence of concentrational depolarization in fluorescein in methyl polymethacrylate. The results of measurements are shown Figs. 1 and 2. The ordinates and abscissae represent the relative yield in $\eta/\eta_0$ and $\log v$, respectively. The radii of the active sphere for fluorescein are found to be identical, in both cases, i. e. for fluorescein in methyl polymethacrylate and in ethyl alcohol within the limits of experimental error. Hence, it may be concluded that in the cases under consideration the medium does not appreciably influence quenching resulting from energy transfer from the excited molecule to the quencher.

Measuring the yield of quenching of fluorescein solutions in methyl polymethacrylate was rendered extremely difficult by their low yield as

compared with that of rhodamin B in methyl methacrylate and of fluorescein in ethyl alcohol.

In order to explain the rather considerable and systematic divergences between the experimental results and the theoretical curve at

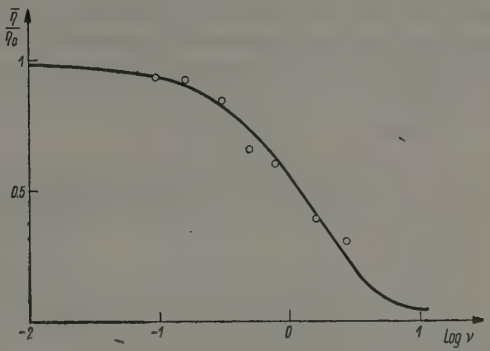


Fig. 1. Relative fluorescence yield of rhodamin B quenched fluorescein in methyl polymethacrylate *versus* log$\nu$.
—— — theoretical curve
⊙ — experimental points

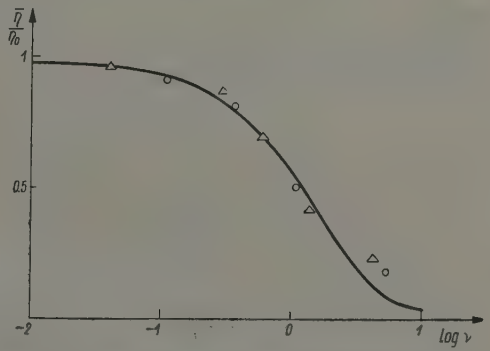


Fig. 2. Relative fluorescence yield *versus* log$\nu$.
—— — theoretical curve
⊙ — experimental points obtained with rhodamin B quenched trypaflavine in ethyl alcohol
△ — experimental points obtained with rhodamin B quenched fluorescein in ethyl alcohol

high concentrations of the quenchers in the case of liquid solutions, an investigation of *pH versus* rhodamin B concentration was carried out. From graphs 3 and 4 it is seen that the variation of *pH* is quite considerable, a fact that must surely be related to the fall in quenching probability. As a result of dissociation of the carboxylic group of rho-

damin B, the absorption spectrum of the quencher at high concentrations is shifted towards longer wave-lengths by approximately 60 Å in all cases under consideration. By an addition of acid it is possible to reduce the shift in the absorption spectrum; the present authors, however, refrained from doing this, having in view the process of polymerization, since the latter is either accelerated or retarded by an addition of extraneous substances. Either eventuality should be avoided as affecting

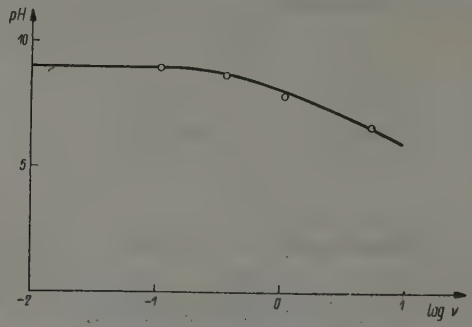


Fig. 3. *pH* of trypaflavine solution in ethyl alcohol *versus* the concentration of rhodamin B

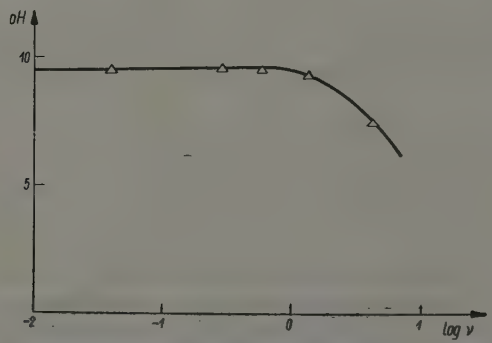


Fig. 4. *pH* of fluorescein solution in ethyl alcohol *versus* the concentration of rhodamin B

the optical homogeneity of the samples (rapid polymerization leads to the formation of bubbles or tensions); conversely, it may happen that the solution fails to polymerize. The effect of concentration of the fluorescent substance on the variation of the active sphere $R_0$ was also investigated. Within the limits of experimental error, a value of $R_0 = 56$ Å was obtained for trypaflavine. Similarly, the radii of the active sphere are identical in the processes of quenching of fluorescein fluorescence with rhodamin B in methyl polymethacrylate and in ethyl alcohol. At

fluorescein concentration $5 \cdot 10^{-4}$ g/cm$^3$ the active radius was $R_0 = 79$ Å, at $10^{-5}$ g/cm$^3$ — $R_0 = 78$ Å.

Undoubtedly, all the mentioned factors act simultaneously to reduce quenching at high concentrations below that predicted by the theoretical curve, which does not account for the effect of the various physico-chemical factors on quenching.

The authors are greatly indebted to Professor A. Jabłoński for suggesting the subject and for his numerous friendly criticisms. The authors express their thanks to Mr A. Kawski for discussing certain matters with them.

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# БЮЛЛЕТЕНЬ
# ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII 1959 ВЫПУСК 5



З. ЛЕВАНДОВСКИЙ, О НЕКОТОРЫХ КЛАССАХ ОДНОЛИСТНЫХ ФУНКЦИЙ, ВВЕДЕННЫХ П. МОНТЕЛЕМ И В. РОГОЗИНСКИМ . . . . . стр. 261–265


Пусть $S$, $S_0$, $S_p$ ($p=1,2,\ldots$) — классы голоморфных и однолистных функций, определенных в круге $|z|<1$, нормированные следующим образом: 1) если $f(z)\in S$, то $f(0)=0$, $f'(0)=1$; 2) если $f(z)\in S_0$, то $f(0)=0$, $f(z_0)=z_0$; 3) если $f(z)\in S_p$, то $f(0)=0$, $f^{(p)}(z_0)=1$, где $z_0$ — фиксированное для данного класса число такое, что $0\leqslant|z_0|<1$. Через $S^*$, $S_0^*$, $S_p^*$ ($p=1,2,\ldots$) обозначим соответственно подклассы классов $S$, $S_0$, $S_p$, состоящие из функций, отображающих круг $|z|<1$ на области, звездообразные относительно начала координат.

В работе приводятся следующие теоремы:

Теорема 1. *Образ круга* $|z|<1$, *полученный при помощи любой функции* $f(z)\in S_0$, *покрывает круг*

$$|z|<\tfrac{1}{4}(1-|z_0|)^2.$$

Теорема 2. *Образ круга* $|z|<1$, *полученный при помощи любой функции* $f(z)\in S_1$, *покрывает круг*

$$|z|<\frac{1}{4}\,\frac{(1-|z_0|)^3}{1+|z_0|}.$$

Кроме того, приводятся Теоремы 1′ и 2′, содержащие аналогичные результаты для подклассов выпуклых функций классов $S_0$ и $S_1$.

Теорема 3. *Если любая функция* $f(z)\in S$, $f(z)=z+a_2z^2+\ldots$ *удовлетворяет гипотезе Вибербаха*: $|a_n|\leqslant n$, $n=2,3,\ldots,p$, *то образ круга* $|z|<1$, *полученный при помощи произвольной функции класса* $S_p$ *покрывает круг*

$$|z|<\frac{1}{4}\,\frac{(1-|z_0|)^{p+2}}{p!(p+|z_0|)}.$$

Во всех этих теоремах приводятся экстремальные функции.

Наконец, приводится без доказательства следующая

Теорема 4. *Множество значений функционала* $\sqrt{\frac{z}{\varphi(z)}}$ *при фиксированном* $z$, $|z|<1$, *в классе* $S_0^*$ *есть замкнутый круг*

$$\left|\sqrt{\frac{z}{\varphi(z)}}-\frac{1-z\bar{z}_0}{1-|z_0|^2}\right|\leqslant\frac{|z-z_0|}{1-|z_0|^2}.$$

Отсюда, используя зависимость между классами $S_0^*$ и $S^*$ выводится следующая

Теорема 4′. *Множество значений функционала* $\sqrt{\frac{\varphi(z)}{z}}$, *при фиксированном* $z$, $|z|<1$, *в классе* $S_0^*$ *есть замкнутый круг*

$$\left|\sqrt{\frac{\varphi(z)}{z}}-\frac{1-\bar{z}z_0'}{1-|z|^2}\right|\leqslant\frac{|z-z_0|}{1-|z|^2}.$$

На основании Теорем 4 и 4′ легко выводятся точные оценки модуля функции в классе $S_0^*$.

Если $\varphi(z)\in S_0^*$, то $|z|\left(\frac{1-|z_0|^2}{|z-z_0|+|1-z\bar{z}_0|}\right)^2\leqslant|\varphi(z)|\leqslant|z|\left(\frac{|z-z_0|+|1-z_0\bar{z}|}{1-|z|^2}\right)^2$.

Теорема 1 является обобщением аналогичной теоремы В. Рогозинского [6]. В. Рогозинский доказал теорему 1 только для класса $S_0^*\subset S_0$. Теорема 2 для класса $S_1^*$, при дополнительном предположении $|z_0|<\frac{1}{10}$, была получена ранее М. Бернацким [2].

С. ТРЫБУЛА, ОБ ОЦЕНКЕ СНИЗУ ВЕРОЯТНОСТНЫХ МОМЕНТОВ . . . . . стр. 267–269

Пусть $X$ случайная величина с абсолютно непрерывным распределением, для которой $k$-ый центральный абсолютный момент $m_k$ конечен. Через $f(x)$ обозначим плотность величины $X$ и положим

$$\pi_n=\int_{-\infty}^{+\infty}f^n(x)\,dx.$$

Целью работы является найти infimum функционала $\pi_n$ в зависимости от $m_k$. Полученный результат выражается неравенством

$$\pi_n^k m_k^{n-1}\geqslant\frac{(k_n)^k(n-1)^{n-1}}{(k_n+n-1)^{k+n-1}}\left[\frac{k}{2B\left(\frac{1}{k},\frac{n}{n-1}\right)}\right]^{k(n-1)}, \tag{1}$$

где $B(\alpha,\beta)$ есть функция бета. Равенство в формуле (1) имеет место только тогда, когда

$$f_0(x)=\begin{cases}c(1-|ax|^k)^{\frac{1}{n-1}}, & \text{для } -1/a\leqslant x\leqslant 1/a\ (a>0)\\ 0, & \text{для остальных } x.\end{cases}$$

А. ГРАНАС, ТЕОРЕМА ОБ АНТИПОДАХ И ТЕОРЕМЫ О НЕПОДВИЖНЫХ ТОЧКАХ ДЛЯ ОДНОГО КЛАССА МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ В БАНАХОВЫХ ПРОСТРАНСТВАХ . . . . . стр. 271—275

В работе рассматриваются вполне непрерывные многозначные векторные поля в банаховом пространстве. Приводится теорема об антиподах, а также некоторые другие обобщения теорем о неподвижных точках теории Лерэ-Шаудера на случай многозначных отображений. Значениями рассматриваемых в работе многозначных функций являются выпуклые множества в банаховом пространстве.

А. ГРАНАС и Я. В. ЯВОРОВСКИЙ, НЕКОТОРЫЕ ТЕОРЕМЫ О МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЯХ ПОДМНОЖЕСТВ ЕВКЛИДОВЫХ ПРОСТРАНСТВ . . . . . стр. 277—283

В работе рассматриваются многозначные ациклические векторные поля на сфере евклидова пространства. Приводятся определение и свойства кронекеровской характеристики для неисчезающих многозначных векторных полей. При помощи этого понятия обобщается на случай многозначных ациклических отображений классическая теорема об инвариантности области, а также теорема К. Борсука об $\varepsilon$-отображениях евклидовых пространств.

В доказательстве используется, доказываемый в работе, один вариант теоремы К. Борсука об антиподах для многозначных отображений.

М. СУФФЧИНСКИЙ, НУМЕРИЧЕСКИЕ ЗНАЧЕНИЯ ДВУЦЕНТРОВЫХ ИНТЕГРАЛОВ ДЛЯ ЭЛЕКТРОНОВ $3d$ В ПЕРЕХОДНЫХ МЕТАЛЛАХ . . . . . стр. 285—287

Метод тесно связанных электронов хорошо применим для определения полосы $3d$ в переходных металлах. В настоящей работе даются значения двуцентровых интегралов, приводимых в числовых расчетах этого метода.

При исчислении значений этих интегралов необходимо знание атомных функций для состояний $3d$. Радиальная часть этих функций $P_A(r)$ была апроксимирована аналитическим выражением типа Лёвдина, содержащим показательные функции. Наименьший показатель может быть подобран различным образом при апроксимировании этой функции. Доказано, что значения двуцентровых интегралов сильно зависят от подбора наименьшего показателя.

Б. КУХОВИЧ, ПОЛЯРИЗАЦИЯ $\beta$-ЧАСТИЦ В ПЕРВЫХ ЗАПРЕЩЕННЫХ ПЕРЕХОДАХ С ИЗМЕНЕНИЕМ СПИНА $\Delta J = \pm 1$ . . . стр. 289—293

В работе приводятся формулы для поляризации $\beta$-частицы в первых запрещенных переходах при предположении, что за $\beta$-распад ответственны векторное и аксиально-векторное взаимодействия. Приведенные формулы применяемы в опытах, когда измерению подвержена поляризация $\beta$-частицы совместно с угловой корреляцией электрон-нейтрино.

М. ФРОНЦКОВЯК, Я. ГЖИВАЧ, И. ГЕЛЬДТ, АНИЗОТРОПИЯ ЭМИССИИ ФОТОЛЮМИНЕСЦЕНЦИИ АКРИДИНОВОГО ЖЕЛТОГО В ПРЕДВОЗБУЖДЕННОМ СОСТОЯНИИ . . . . . стр. 295–300

Исследованы изменения анизотропии эмиссии фотолюминесценции акридинового желтого в желатине для разной длины волн внутри эмиссионной полосы в зависимости от времени предвозбуждения.

По мере роста времени предвозбуждения анизотропия эмиссии снижается. Изменения анизотропии эмиссии различны для разной длины волн внутри эмиссионной полосы. Наблюдаемое явление можно объяснить наложением полос люминесценции деформированных и недеформированных центров, которые образуются вследствие сильного предвозбуждения.

И. ГЛОВАЦКИЙ и Р. ПОГОСКИЙ, ТУШЕНИЕ ФОТОЛЮМИНЕСЦЕНЦИИ ЖИДКИХ И ЖЕСТКИХ РАСТВОРОВ . . . . . стр. 301–305

Проверена экспериментально теория тушения фотолюминесценции чужими поглощающими субстанциями, предложенная А. Яблонским.